Bridgeland Stability Conditions, Stability of the Restricted Bundle, Brill-Noether Theory and Mukai's Program

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Abstract

In [Bri07], Bridgeland introduced the notion of stability conditions on the bounded derived category $\mathcal{D}(X)$ of coherent sheaves on an algebraic variety X. This topic is originally inspired by concepts in string theory and mathematical physics and has many interesting applications in algebraic geometry.

In the first part of the thesis, we provide a direct proof of an important result in [Bri08, BMS16] which states there is a two dimensional family of weak Bridgeland stability conditions on the bounded derived category $\mathcal{D}(X)$ of coherent sheaves on a variety X.

As a first application of this result, we prove an effective restriction theorem which provides sufficient conditions on a stable locally free sheaf on a projective variety such that its restriction to a hypersurface remains stable. Secondly, we extend and complete Mukai's program to reconstruct a K3 surface from a curve on that surface. We show that the K3 surface containing the curve can be obtained uniquely as a Fourier-Mukai partner of a suitable Brill-Noether locus of vector bundles on the curve.

Lay Summary

Algebraic geometry classically studies objects called varieties which locally look like the zero set of a collection of multivariate polynomials. Modern algebraic geometry is built on applying the abstract algebraic techniques to study geometrical problems about these sets of zeros.

Derived categories are a method to translate the information about these geometric objects into the algebraic concepts. In 2007, Bridgeland introduced the new notion of stability conditions on derived categories. This topic originally inspired by concepts in string theory and mathematical physics. In this thesis, we concentrate on the applications of stability condition in classical algebraic geometry.

There are various questions that one can think about them via this new method. For instance, if we define an object on a surface, which properties of this object will be preserved when we restrict it to a curve on the surface? Or conversely, which information about the surface can be obtained from a curve on the surface? Can we uniquely reconstruct the surface by using the information on the curve and some extra assumptions about the surface?

Contents

Chapter 1

Introduction

Let X be a smooth projective variety. In [Bri07], Bridgeland defined stability conditions on the bounded derived category $\mathcal{D}(X)$ of coherent sheaves on an algebraic variety X. This is a very important topic because of its applications in various subjects, including Donaldson-Thomas invariants [Tod14], Birational geometry of moduli spaces [ABCH13, BM14], Brill-Noether theory [Bay16b] and derived category [BB17].

A Bridgeland stability condition is a pair $\sigma = \big(Z, \mathcal{A}\big)$ consisting of an abelian subcategory $\mathcal{A} \subset \mathcal{D}(X)$ and a linear function Z that associates to any element of $\mathcal{A}$ a complex number. An object of $\mathcal{A}$ is called stable if the corresponding numbers for all of its subobjects have smaller phases. The space of Bridgeland stability conditions on the bounded derived category $\mathcal{D}(X)$ enjoys some remarkable properties. First of all, the space of stability conditions is a complex manifold $\mathrm{Stab}(X)$ with its natural topology. Secondly, the space of stability conditions admits a wall and chamber decomposition for any fixed object E in $\mathcal{D}(X)$, which means stability of E is unchanged when the stability condition varies within a chamber and it switches from stability to instability, or vice versa when we cross a wall. Finally, there exists a chamber, called Gieseker chamber, where the classical notion of Gieseker-stability for coherent sheaves coincides with the Bridgeland stability.

In general, it is not known how we can construct Bridgeland stability conditions on the bounded derived category of coherent sheaves. However, for some applications, we only need the notion of "weak" stability condition. In [Bri08, BMS16], the authors prove the following important result.

Theorem 2.1.1 ([Bri08, BMS16])**.** *There is a continuous family of weak Bridgeland stability conditions on $\mathcal{D}(X)$, parametrised by $\mathbb{R} \times \mathbb{R}^{>0}$.*

In chapter 2, we provide a direct proof for Theorem 2.1.1, without using the general deformation property of Bridgeland stability conditions. The idea is to first consider the weak Bridgeland stability conditions corresponding to rational points in $\mathbb{Q} \times \mathbb{Q}^{>0}$. Then we show that by fixing one of the parameters and changing the other one, we get a continuous path of weak Bridgeland stability conditions .

Wall-crossing with respect to weak Bridgeland stability conditions provides a direct way to study slope stability of torsion free sheaves. In Chapter 3, we study sufficient conditions on a slope stable vector bundle such that its restriction to a hypersurface remains stable. We first reprove one of the Langer's effective restriction theorems.

Theorem 3.1.1 ([Lan04])**.** *Let X be a smooth projective complex variety of dimension $n \geq 2$. Let H be an ample divisor on X and let E be a μ_H-stable vector bundle of rank* *rk* ≥ 2 *on* X. *Take an integer* $l > l_0$ *where* l_0 *depends on the discriminant of* E *and its rank, then for any divisor* $C \in |lH|$, *the restriction* $E|_C$ *is* $\mu_{H|_C}$*-stable.*

To prove Theorem 3.1.1, it is enough to find a weak Bridgeland stability conditions ν such that both objects E and $E(-lH)[1]$ are ν-stable of the same phase. Then $i_*E|_C$ for a divisor $C \in |lH|$ with embedding $i\colon C \hookrightarrow X$, fits into the short exact sequence

$$E \hookrightarrow i_*E|_C \twoheadrightarrow E(-lH)[1].$$

Thus $i_*E|_C$ is also ν-semistable and a general argument implies that the restriction $E|_C$ is $\mu_{H|_C}$-stable.

One may apply the same strategy to get stronger restriction theorems in special cases. For instance, we can prove the stability of the restriction of Lazarsfeld-Mukai bundles associated to line bundles on curves in K3 surfaces, which gives many new counterexamples to Mercat's conjecture [Fey16].

In Chapter 4, we study Mukai's program to answer the following interesting question: can we uniquely reconstruct a K3 surface from a curve on that surface? To be more precise, we need to consider some moduli spaces. Let $\mathcal{F}_g$ be the moduli space of polarised K3 surfaces (X, H) where H is a primitive ample line bundle on X and $H^2 = 2g - 2$. Let $\mathcal{P}_g$ be the moduli space of pairs (X, C) such that $(X, H) \in \mathcal{F}_g$ and C is a smooth curve in the linear system $|H|$. Finally, let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g. The space $\mathcal{P}_g$ has natural projections to $\mathcal{F}_g$ and $\mathcal{M}_g$ which we denote by ϕ_g and m_g, respectively;

$$\begin{tikzcd}
& \mathcal{P}_g \arrow[dl, "m_g"'] \arrow[dr, "\phi_g"] & \\
\mathcal{M}_g & & \mathcal{F}_g
\end{tikzcd}$$

Ciliberto, Lopez and Miranda [CLM93] proved that for $g \geq 11$ and $g \neq 12$, the map m_g is birational onto its image, thus we can think of the rational inverse of m_g. In [Muk01], Mukai introduced a geometric program to find the rational inverse of m_g where $g = 2s + 1$ and $s \geq 5$ odd. Let C be a general curve on the image of m_g. His idea is to consider the Brill-Noether locus BN of stable rank 2-vector bundles on the curve C with canonical determinant and possessing high enough number of global sections. Then he conjectured that the Brill-Noether locus BN is a K3 surface and the K3 surface containing the curve C can be obtained uniquely as a Fourier-Mukai partner of the Brill-Noether locus BN. In Chapter 4, we generalise Mukai's program to any genus $g \geq 13$ or $g = 11$, by considering the Brill-Noether loci of higher rank vector bundles on the curve C.

Theorem 4.1.1. *Let* (X, H) *be a polarised K3 surface with Pic*$(X) = \mathbb{Z}.H$. *Let* C *be any curve in the linear system* $|H|$ *of genus* $g = 11$ *or* $g \geq 13$. *Then* X *is the unique K3 surface of Picard rank one and genus* g *containing* C, *and can be reconstructed as a Fourier-Mukai partner of a certain Brill-Noether locus BN of vector bundles on* C.

The key point is that any vector bundle in the Brill-Noether locus BN is the restriction of a stable vector bundle on the surface X. To prove this result, we introduce a new upper bound on the number of global sections of objects in $\mathcal{D}(X)$, in terms of the perimeter of their Harder-Narasimhan polygons, see Proposition 4.3.4.

Chapter 2

Bridgeland Stability Conditions

2.1 Introduction

In this chapter, we first give a brief review of the notion of a (weak) Bridgeland stability condition on the bounded derived category $\mathcal{D}(X)$ of coherent sheaves on a smooth projective variety X. Then we provide a direct proof of the following result which we will make more precise in Theorem 2.3.1.

Theorem 2.1.1 ([Bri08, BMS16])**.** *There is a continuous family of weak Bridgeland stability conditions on $\mathcal{D}(X)$, parametrised by $\mathbb{R} \times \mathbb{R}^{>0}$.*

In the proof, we do not use the general deformation property of (weak) Bridgeland stability conditions. The first step is to apply the properties of slope stable sheaves, to show that there is a weak stability condition $\nu_{(b,a)}$ for any pair $(b,a) \in \mathbb{Q} \times \mathbb{R}^{>0}$. Then we consider the function

$$\gamma \colon t \in [0,1] \cap \mathbb{Q} \mapsto \nu_{(b+t,a)}$$

and prove that it is a continuous path of weak stability conditions, where $b \in \mathbb{Q}$. This implies that $\nu_{(b,a)}$ is a weak stability condition for any pair $(b,a) \in \mathbb{R} \times \mathbb{R}^{>0}$. Finally, we show that the function

$$\gamma' \colon t \in [0,1] \mapsto \nu_{(b,a+t)}$$

gives also a continuous path of weak stability conditions, which leads to Theorem 2.1.1.

2.2 Stability conditions on derived categories

In this section, we recall the notion of (weak) Bridgeland stability conditions on the bounded derived category of coherent sheaves.

Let X be a projective scheme over $\mathbb{C}$ of dimension $n \geq 1$. Recall that the Euler characteristic of a coherent sheaf E on X is

$$\chi(E) = \sum_i (-1)^i \dim_{\mathbb{C}} H^i(X, E).$$

If we fix an ample line bundle H on X, then the Hilbert polynomial $P(E,m)$ is given by

$$m \mapsto \chi(E \otimes H^m).$$

The Hilbert polynomial can be uniquely written in the form

$$P(E,m) = \sum_{i=0}^{\dim(E)} \alpha_i(E)\frac{m^i}{i!}$$

with integral coefficients $\alpha_i(E)$ $(i = 0, ..., \dim(E))$ [HL10]. The reduced Hilbert polynomial $p(E,m)$ of a coherent sheaf $E \neq 0$ of dimension d is defined by

$$p(E,m) = \frac{P(E,m)}{\alpha_d(E)}.$$

Definition 2.2.1. We say that a torsion free sheaf E on X is μ_H-(semi)stable if for all proper non-trivial subsheaves $F \subset E$, we have

$$\frac{\alpha_{\dim(X)-1}(F)}{\alpha_{\dim(X)}(F)} < (\leq) \frac{\alpha_{\dim(X)-1}(E)}{\alpha_{\dim(X)}(E)}.$$

We always assume X is a smooth projective complex variety. We denote by $\mathcal{D}(X) = \mathcal{D}^b\mathrm{Coh}(X)$ the bounded derived category of coherent sheaves on X. For an object $E \in \mathcal{D}(X)$, we consider the Chern character $\mathrm{ch}(E) = \big(\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E), ...\big)$. The slope of a coherent sheaf E with positive rank $\mathrm{ch}_0(E) > 0$ is defined as

$$\mu_H(E) \coloneqq \frac{H^{n-1}.\mathrm{ch}_1(E)}{H^n \mathrm{ch}_0(E)},$$

and if $\mathrm{ch}_0(E) = 0$, define $\mu_H(E) \coloneqq +\infty$. Definition 2.2.1 implies that a torsion free sheaf E is μ_H-(semi)stable if and only if for any non-trivial subsheaf $F \subset E$, we have $\mu_H(F) < (\leq)\, \mu_H(E)$. Any torsion free sheaf E has a unique Harder-Narasimhan (HN) filtration with respect to μ_H-stability. It is a sequence of objects in $\mathrm{Coh}(X)$,

$$0 = E_0 \subset E_1 \subset E_2 \subset ... \subset E_n = E$$

where E_i/E_{i-1} is μ_H-semistable for $1 \leq i \leq n$, and

$$\mu_H^+(E) \coloneqq \mu_H(E_1/E_0) > \mu_H(E_2/E_1) > .... > \mu_H(E_n/E_{n-1}) =: \mu_H^-(E).$$

To define a notion of stability, we may consider other abelian subcategories of $\mathcal{D}(X)$ which satisfy some nice properties.

Definition 2.2.2 ([BlBD82, Bri08])**.** The heart of a bounded t-structure on $\mathcal{D}(X)$ is a full additive subcategory $\mathcal{A} \subset \mathcal{D}(X)$ such that

(a) if A and B are objects of $\mathcal{A}$, then $\mathrm{Hom}(A, B[k]) = 0$ for $k < 0$.
(b) for every non-zero object $E \in \mathcal{D}(X)$, there are integers $m < n$ and a sequence of distinguished triangles

$$\begin{tikzcd}
0 = E_m \arrow[rr] & & E_{m+1} \arrow[rr] \arrow[dl] & & E_{m+2} \arrow[r] \arrow[dl] & \dots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_{m+1} \arrow[ul, dashed] & & A_{m+2} \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_i[i] \in \mathcal{A}$ for all i.

The objects $A_i[i]$ of $\mathcal{A}$ are called the cohomology objects of E in the bounded t-structure. We write them as $\mathcal{H}^i_{\mathcal{A}}(E) = A_i[i]$.

Note that a bounded t-structure on $\mathcal{D}(X)$ is uniquely determined by its heart. One can construct many non-trivial t-structures using tilting at a torsion pair [HRS96].

Definition 2.2.3. A torsion pair in an abelian category $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full additive subcategories such that

(a) $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$.
(b) For all $E \in \mathcal{A}$, there exists a short exact sequence

$$0 \to T \to E \to F \to 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Example 2.2.4. One of the main examples of a torsion pair is for $\mathcal{A} = \mathrm{Coh}(X)$, where $\mathcal{T}$ is the full subcategory of torsion sheaves and $\mathcal{F}$ is the full subcategory of torsion free sheaves.

The existence of HN filtration with respect to μ_H-stability implies that for any $b \in \mathbb{R}$, we have the torsion pair $(\mathcal{T}^b, \mathcal{F}^b)$ in $\mathrm{Coh}(X)$, where

$$\mathcal{T}^b \coloneqq \{E \in \mathrm{Coh}(X) \colon \mu_H^-(E) > b\} \text{ and } \mathcal{F}^b \coloneqq \{E \in \mathrm{Coh}(X) \colon \mu_H^+(E) \leq b\}.$$

Proposition 2.2.5 ([Bri08, HRS96])**.** *The following gives the heart of a bounded t-structure on $\mathcal{D}^b(X)$:*

$$\begin{aligned} \mathcal{A}(b) &= \left\langle \mathcal{F}^b[1], \mathcal{T}^b \right\rangle \\ &= \left\{ E : \ H^0(E) \in \mathcal{T}^b, H^{-1}(E) \in \mathcal{F}^b, H^i(E) = 0 \text{ for } i \neq 0, -1 \right\}, \end{aligned}$$

where $\langle - \rangle$ denotes the extension-closure.

The Grothendieck group $K(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the quotient of the free abelian group generated by its objects by the relation $[B] = [B'] + [B'']$ for any short exact sequence $B' \hookrightarrow B \twoheadrightarrow B''$. For the abelian category $\mathcal{A} = \mathrm{Coh}(X)$, the Grothendieck group $K(X) \coloneqq K(\mathrm{Coh}(X))$ is generated by the classes of vector bundles $[F]$ on the variety X, up to the relation defined above. For instance, the Grothendieck group $K(\mathbb{P}^n)$ is generated by $\{\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), ..., \mathcal{O}_{\mathbb{P}^n}(n)\}$.

We always fix a finite rank lattice Λ with a surjective map

$$v \colon K(\mathcal{A}) \twoheadrightarrow \Lambda. \tag{2.1}$$

In [Bri07], Bridgeland introduced the notion of stability conditions on triangulated categories. Some variant notions also appeared in [Tod10, BMS16].

Definition 2.2.6. A weak stability function on an abelian category $\mathcal{A}$ is a group homomorphism $Z \colon \Lambda \to \mathbb{C}$ such that for any object $E \in \mathcal{A}$,

$$Z(v(E)) = m(v(E)) \exp(i\pi\phi(v(E))) \quad \text{where} \quad m(v(E)) \geq 0 \text{ and } 0 < \phi(v(E)) \leq 1.$$

If for any non-trivial object E, we have $Z(v(E)) \neq 0$, the homomorphism Z is called a stability function on $\mathcal{A}$.

If $Z(v(E)) = 0$ for a non-trivial object $E \in \mathcal{A}$, then we define $\phi(v(E)) = 1$. The real number $\phi(v(E)) \in (0, 1]$ is called the phase of the object E. We will abuse notations and write $Z(E)$ and $\phi(E)$ instead of $Z(v(E))$ and $\phi(v(E))$.

Definition 2.2.7. A non-zero object $E \in \mathcal{A}$ is said to be (semi)stable with respect to a stability function Z if

$$0 \neq E' \subset E \quad \Rightarrow \quad \phi(E') < (\leq)\, \phi(E).$$

We say that the stability function Z satisfies the Harder-Narasimhan property if every non-zero object $E \in \mathcal{A}$ has a finite filtration

$$0 = E_0 \subset E_1 \subset ... \subset E_{n-1} \subset E_n = E$$

whose factors $F_i = E_i/E_{i-1}$ are semistable and

$$\phi^+(E) \coloneqq \phi(F_1) > \phi(F_2) > ... > \phi(F_n) =: \phi^-(E).$$

Definition 2.2.8. A (weak) stability function Z on an abelian category $\mathcal{A}$ satisfies the support property if there exists a quadratic form Q on the vector space $\Lambda_{\mathbb{R}}$ such that

(a) the kernel of Z is negative definite with respect to Q, and
(b) for any semistable object $E \in \mathcal{A}$ with respect to Z, we have $Q(v(E)) \geq 0$.

See [BMS16, Appendix A] for other equivalent definitions of the support property.

Definition 2.2.9 ([Bri07, BMS16])**.** A (weak) stability condition on $\mathcal{D}(X)$ is a pair $\nu = \big(Z, \mathcal{A}\big)$ where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}(X)$ and Z is a (weak) stability function on A which satisfies the Harder-Narasimhan property and the support property.

An object $E \in \mathcal{D}(X)$ is said to be ν-(semi)stable if some shift $E[k]$ is contained in the abelian category $\mathcal{A}$ and the object $E[k]$ is (semi)stable with respect to the function Z.

Definition 2.2.10 ([Bri07])**.** A slicing $\mathcal{P}$ of a triangulated category $\mathcal{D}$ consists of full additive subcategories $\mathcal{P}(\phi)$ for each $\phi \in \mathbb{R}$ satisfying the following axioms:

(a) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,
(b) if $\phi_1 > \phi_2$ and $A_i \in \mathcal{P}(\phi_i)$ then $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$,
(c) for each non-zero object $E \in \mathcal{D}$ there is a finite sequence of real numbers

$$\phi^+(E) \coloneqq \phi_1 > \phi_2 > ... > \phi_n =: \phi^-(E)$$

and a collection of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \dots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& F_1 \arrow[ul, dashed] & & F_2 \arrow[ul, dashed] & & & & F_n \arrow[ul, dashed] &
\end{tikzcd}$$

with $F_i \in \mathcal{P}(\phi_i)$ for all i.

Any weak stability condition $\nu = \big(Z, \mathcal{A}\big)$ defines a slicing $\mathcal{P}_\nu$ of $\mathcal{D}(X)$ as follows: for each $\phi \in (0, 1]$, let $\mathcal{P}_\nu(\phi)$ be the full additive subcategory of $\mathcal{D}(X)$ consisting of semistable objects with phase ϕ, together with zero. The part (a) of Definition determines $\mathcal{P}_\nu(\phi)$ for all $\phi \in \mathbb{R}$. For any nonzero object $E \in \mathcal{D}(X)$, a filtration as in part (c) can be obtained by combining the decompositions of the heart $\mathcal{A}$ of a bounded t-structure with the Harder-Narasimhan filtrations of objects in $\mathcal{A}$. For any object $E \in \mathcal{D}(X)$, we define $\phi_\nu^+(E) \coloneqq \phi(F_1)$ and $\phi_\nu^-(E) \coloneqq \phi(F_n)$.

We denote by $\mathrm{WStab}_\Lambda(X)$ the set of weak stability conditions $\nu = \big(Z, \mathcal{A}\big)$ on $\mathcal{D}(X)$ where the stability functions Z factors via the surjection $v \colon K(X) \twoheadrightarrow \Lambda$. The set of slicings of $\mathcal{D}(X)$ is denoted by $\mathrm{Slice}(X)$. In [Bri07], Bridgeland introduced a generalised metric on $\mathrm{Slice}(X)$ as follows:

$$d(\mathcal{P}, \mathcal{Q}) = \sup_{0 \neq E \in \mathcal{D}(X)} \Big\{ \big|\phi^+_{\mathcal{P}}(E) - \phi^+_{\mathcal{Q}}(E)\big| , \big|\phi^-_{\mathcal{P}}(E) - \phi^-_{\mathcal{Q}}(E)\big| \Big\} \in [0, \infty].$$

The topology on $\mathrm{WStab}_\Lambda(X)$ is the coarsest topology such that both maps

$$\mathrm{WStab}_\Lambda(X) \to \mathrm{Slice}(X), \quad \nu = (Z, \mathcal{A}) \to \mathcal{P}_\nu$$

$$\mathrm{WStab}_\Lambda(X) \to \mathrm{Hom}(\Lambda, \mathbb{C}) \quad \nu = (Z, \mathcal{A}) \to Z$$

are continuous, see [Bri07, Bri08, BMS16, Bay16a] for more details.

2.3 Two dimensional family of weak stability conditions

In this section, we show that there is a 2-dimensional family of weak stability conditions on the derived category of coherent sheaves on a smooth projective variety.

Let X be a smooth projective complex variety of dimension $n \geq 2$ and let H be a primitive ample line bundle on X. Consider the homomorphism

$$v_H \colon K(X) \twoheadrightarrow \mathbb{Z} \,\oplus\, \mathbb{Z} \,\oplus\, \frac{1}{2}\mathbb{Z}$$

defined by $v_H(E) = \big(H^n \mathrm{ch}_0(E), H^{n-1}\mathrm{ch}_1(E), H^{n-2}\mathrm{ch}_2(E)\big)$. Let Λ be the image of v_H. The bilinear form

$$\big\langle (r, c, h), (r', c', h') \big\rangle = cc' - rh' - hr'$$

makes Λ into a lattice of signature $(2, 1)$. Corresponding to any pair $(b, a) \in \mathbb{R} \times \mathbb{R}^{>0}$, we consider the pair $\nu_{(b,a)} = \big(Z_{(b,a)}, \mathcal{A}(b)\big)$ of the function

$$Z_{(b,a)} \colon \Lambda \to \mathbb{C} \;\;,\;\; Z_{(b,a)}(r, c, h) = \left\langle (r, c, h), \left(1, b, \frac{b^2}{2} - \frac{a^2}{2}\right) \right\rangle + i\big\langle (r, c, h), (0, 1, b) \big\rangle,$$

and the heart $\mathcal{A}(b) = \langle \mathcal{F}^b[1], \mathcal{T}^b \rangle$, where

$$\mathcal{T}^b \coloneqq \{E \in \mathrm{Coh}(X) \colon \mu_H^-(E) > b\} \;\text{ and }\; \mathcal{F}^b \coloneqq \{E \in \mathrm{Coh}(X) \colon \mu_H^+(E) \leq b\}.$$

Note that the function $Z_{(b,a)}$, up to the action of $\mathrm{GL}^+(2, \mathbb{R})$, is the same as the stability function defined in [Bri08, Section 6]. The final goal of this section is to provide a direct proof of the following result.

Theorem 2.3.1 ([Bri08, BMS16])**.** *There is a continuous family of weak stability conditions parametrised by $\mathbb{R} \times \mathbb{R}^{>0}$ given by*

$$(b, a) \to \nu_{(b,a)} = \big(Z_{(b,a)}, \mathcal{A}(b)\big).$$

The discriminant of an object $E \in \mathcal{D}(X)$ with respect to H is defined by

$$\overline{\Delta}_H(E) = \big(H^{n-1}\mathrm{ch}_1(E)\big)^2 - 2H^n\mathrm{ch}_0(E)H^{n-2}\mathrm{ch}_2(E).$$

This defines the quadratic form

$$Q(r,c,h) \coloneqq c^2 - 2rh$$

on the vector space $\Lambda_{\mathbb{R}}$, which is the quadratic form corresponding to the bilinear form $\langle -,- \rangle$. The kernel of $Z_{(b,a)}$ which is spanned by $(1, b, b^2/2 + a^2/2)$ in $\Lambda_{\mathbb{R}}$, is negative definite with respect to Q. The Bogomolov inequality [HL10, Theorem 7.3.1], and Hodge index theorem imply that for any μ_H-semistable sheaf E, we have $\overline{\Delta}_H(E) \geq 0$.

Lemma 2.3.2. *The function $Z_{(b,a)}$ is a weak stability function on the heart $\mathcal{A}(b)$ for any pair $(b,a) \in \mathbb{R} \times \mathbb{R}^{>0}$.*

Proof. It is enough to show that for a μ_H-stable sheaf E or its shift $E[1]$ in $\mathcal{A}(b)$, the image lies in the upper half plane or non-positive real line.

If $E \in \mathcal{A}(b)$ is a torsion free sheaf, then by definition $\mu_H(E) > b$ and imaginary part $\mathrm{Im}[Z_{(b,w)}(E)] > 0$ is positive. If E is a torsion sheaf and $\mathrm{ch}_1(E) \neq 0$, then $\mathrm{ch}_1(E).H^{n-1} > 0$ and again the imaginary part is positive. If $\mathrm{ch}_1(E) = 0$ for a torsion sheaf E, then $\mathrm{ch}_2(E).H^{n-2} \geq 0$ which means the real part $\mathrm{Re}[Z_{(b,w)}(E)] \leq 0$ is non-positive. Similarly, if E is a μ_H-stable torsion free sheaf with $\mu_H(E) < b$, the complex number $Z_{(b,w)}(E[1])$ lies in the upper half plane. If $\mu_H(E) = b$, then

$$\mathrm{Re}\big[Z_{(b,w)}\big(E[1]\big)\big] = -\frac{\overline{\Delta}_H(E)}{2\mathrm{ch}_0(E).H^n} - \mathrm{ch}_0(E).H^n\frac{a^2}{2},$$

which is negative by the Bogomolov inequality, so the claim follows. $\square$

This implies that for any non-zero object $E \in \mathcal{A}(b)$, we have

$$Z(E) = m\exp\big(i\pi\phi_{(b,a)}(E)\big) \quad \text{where} \quad m \geq 0 \;\; \text{and} \;\; 0 < \phi_{(b,a)}(E) \leq 1.$$

If $Z(E) \neq 0$, the phase function is

$$\phi_{(b,a)}(E) = \frac{1}{\pi}\tan^{-1}\left(-\frac{\mathrm{Re}[Z_{(b,a)}(E)]}{\mathrm{Im}[Z_{(b,a)}(E)]}\right) + \frac{1}{2}$$

and if $Z(E) = 0$, we define $\phi_{(b,a)}(E) \coloneqq 1$.

If $v_H(E) = (r,c,h)$ and $r \neq 0$, then

$$\mathrm{Re}[Z_{(b,a)}(E)] = \frac{-1}{2r}(c-br)^2 + \frac{\overline{\Delta}_H(E)}{2r} + \frac{ra^2}{2}, \tag{2.2}$$

and

$$-\frac{\mathrm{Re}[Z_{(b,a)}(E)]}{\mathrm{Im}[Z_{(b,a)}(E)]} = \frac{1}{2}\left(\frac{c}{r} - b\right) - \frac{\overline{\Delta}_H(E)/2r^2 + a^2/2}{c/r - b}. \tag{2.3}$$

Denote by $P \coloneqq \overline{\Delta}_H(E)/2r^2 + a^2/2$. If $x \coloneqq b - c/r \neq 0$, then

$$\frac{d\phi_{(b,a)}(E)}{db} = -\frac{x^2/2 + P}{\pi\big((x^2/2 - P)^2 + x^2\big)}, \tag{2.4}$$

and

$$\frac{d\phi_{(b,a)}(E)}{da} = \frac{ax}{\pi\big((x^2/2 - P)^2 + x^2\big)}. \tag{2.5}$$

If $b \in \mathbb{Q}$, then $\mathrm{Im}[Z_{(b,a)}(E)]$ lies in a discrete group for any object $E \in \mathcal{A}(b)$. Moreover, Lemma 2.3.2 implies that the real part $\mathrm{Re}[Z_{(b,a)}(E)] \leq 0$ if $\mathrm{Im}[Z_{(b,a)}(E)] = 0$, so we have the following result.

Lemma 2.3.3. *[Bri08, Proposition 7.1] There are no infinite sequences of subobjects in $\mathcal{A}(b)$*

$$... \subset E_{i+1} \subset E_i \subset .... \subset E_2 \subset E_1$$

with $\phi_{(b,a)}(E_{i+1}) > \phi_{(b,a)}(E_i)$ for a pair $(b,a) \in \mathbb{Q} \times \mathbb{R}^{>0}$. .

Lemma 2.3.4. *Suppose the objects E_1 and E_2 are in the heart $\mathcal{A}(b)$ and they have the same phase with respect to the stability function $Z_{(b,a)}$ for some $(b,a) \in \mathbb{R} \times \mathbb{R}^{>0}$. If $Q(v_H(E_i)) \geq 0$ for $i = 1, 2$, then $Q(v_H(E_1) + v_H(E_2)) \geq 0$.*

Proof. The following is a set of basis vectors in the vector space $\Lambda_{\mathbb{R}} = \mathbb{R}^3$,

$$\overline{v}_1 = (0, 1, b)\ ,\ \ \overline{v}_2 = \frac{1}{a}\left(1, b, \frac{b^2}{2} - \frac{a^2}{2}\right)\ ,\ \ \overline{v}_3 = \frac{1}{a}\left(1, b, \frac{b^2}{2} + \frac{a^2}{2}\right).$$

For any vector $v = x\overline{v}_1 + y\overline{v}_2 + z\overline{v}_3 \in \Lambda_{\mathbb{R}}$, we have $Q(v) = x^2 + y^2 - z^2$. Assume $v_H(E_i) = x_i\overline{v}_1 + y_i\overline{v}_2 + z_i\overline{v}_3$ for $i = 1, 2$. Since $E_1, E_2 \in \mathcal{A}(b)$, we have $x_i \geq 0$. If $x_1 = x_2 = 0$, then clearly the claim holds. Otherwise, $y_1/x_1 = y_2/x_2 =: k$ and

$$(1 + k^2)x_i^2 \geq z_i^2 \quad \Rightarrow (1 + k^2)x_1x_2 \geq |z_1z_2|\,.$$

Therefore,

$$(1 + k^2)(x_1 + x_2)^2 \geq (z_1 + z_2)^2,$$

as we required. □

One can generalise the Bogomolov inequality to $\nu_{(b,a)}$-semistable objects.

Lemma 2.3.5 ([BMS16])**.** *Assume an object $E \in \mathcal{A}(b)$ is $\nu_{(b,a)}$-semistable for a pair $(b,a) \in \mathbb{Q} \times \mathbb{R}^{>0}$. Then $\overline{\Delta}_H(E) \geq 0$.*

Proof. The argument is the same as [BMS16, Theorem 3.5], we provide it for completeness. Write $v_H(E) = (r, c, h)$ and $b = p/q$ for integers p and q with $\gcd(p, q) = 1$. The proof is by induction on $\mathrm{Im}[Z_{(b,a)}(E)] = c - rb$, which is a non-negative number in a discrete group. Start by increasing a. First assume that the object E remains semistable where $a \to \infty$. This is the case when $\mathrm{Im}[Z_{(b,a)}(E)] = 0$ or $1/q$. Then, [BMS16, Lemma 2.7] implies that E is a μ_H-semistable sheaf or $H^{-1}(E)$ is a μ_H-semistable torsion free sheaf and $H^0(E)$ is supported in codimension at least 2. In both cases, the Bogomolov inequality implies that $\overline{\Delta}_H(E) \geq 0$. Now assume the object E gets destabilized when we increase a. Define

$$a^* := \sup\{a \in \mathbb{R} \colon E \text{ is } \nu_{(b,a)}\text{-semistable}\}.$$

We claim that E is $\nu_{(b,a^*)}$-strictly semistable. Note that the equality (2.3) implies that for two objects $F_1, F_2 \in \mathcal{A}(b)$ with $v_H(F_1)/v_H(F_2) \notin \mathbb{R}$, there exists at most one real number $a_0 \in \mathbb{R}$ such that $\phi_{(b,a_0)}(F_1) = \phi_{(b,a_0)}(F_2) < 1$.

Let F be a subobject of E such that $\phi_{(b,a_0)}(F) = \phi_{(b,a_0)}(E)$ for some $a_0 \geq a^*$. Then, Lemma 2.3.3 implies that there is a $\nu_{(b,a_0)}$-semistable subobject $E' \subset F$ such that $\phi_{(b,a_0)}(E') \geq \phi_{(b,a_0)}(F)$. There exists a real number $a_1 \leq a_0$ such that $\phi_{(b,a_1)}(E') = \phi_{(b,a_1)}(E)$. The imaginary part of $Z_{(b,a_1)}(E')$ is strictly less than the imaginary part of $Z_{(b,a_1)}(E)$, so the induction assumption gives $\overline{\Delta}_H(E') \geq 0$. Thus to compute a^*, it is enough to consider the subobjects with non-negative discriminant;

$$a^* = \sup\{a \in \mathbb{R} \colon \nexists\, E' \subset E \ \text{ with } \ \phi_{(b,a)}(E') > \phi_{(b,a)}(E) \text{ and } \overline{\Delta}_H(E') \geq 0\}.$$

Define the subset

$$V \coloneqq \{kZ_{(b,a)}(E) \mid a \in [a^*, a^*+1] \text{ and } 0 \leq k \leq 1\} \subset \mathbb{C},$$

which is a bounded area. The number $Z_{(b,a')}(E') \in V$ is in the subset V, for any subobject $E' \subset E$ which has the same phase as E with respect to $Z_{(b,a')}$ for some $a' \in [a^*, a^*+1]$. Since $\overline{\Delta}_H(E') \geq 0$, there are only finitely many possible classes in Λ for the subobject E'. This implies that there is a subobject $E' \subset E$ which has the same phase as E exactly at $Z_{(b,a^*)}$, so E is $\nu_{(b,a^*)}$-strictly semistable. Lemma 2.3.4 and the induction assumption give the result. □

Lemma 2.3.5 implies that the stability function $Z_{(b,a)}$ satisfies the support property, if $b \in \mathbb{Q}$. The next step is to verify HN property. The following Lemma shows that the image of the subobjects lies in a bounded area from the left.

Lemma 2.3.6. *Suppose the object E is in the heart $\mathcal{A}(b)$ for a real number $b \in \mathbb{R}$. There exists a real number $M_b(E)$ such that*

$$M_b(E) < Re[Z_{(b,a)}(F)],$$

for any subobject $F \subset E$ in $\mathcal{A}(b)$ and any $a \in \mathbb{R}^{>0}$.

Proof. If the claim holds for two objects E_1 and E_2 in $\mathcal{A}(b)$, then it holds for any extension E

$$E_1 \hookrightarrow E \twoheadrightarrow E_2.$$

Indeed, for any subobject $E' \subset E$, we have $E' \cap E_1 \subset E_1$ and $E'/E' \cap E_1 \subset E_2$, so

$$E' \cap E_1 \hookrightarrow E' \twoheadrightarrow E'/E' \cap E_1.$$

This implies

$$\mathrm{Re}[Z_{(b,a)}(E')] = \mathrm{Re}[Z_{(b,a)}(E' \cap E_1)] + \mathrm{Re}[Z_{(b,a)}(E'/E' \cap E_1)] > M_b(E_1) + M_b(E_2).$$

One can define $M_b(E) \coloneqq M_b(E_1) + M_b(E_2)$. Therefore, it is enough to prove the claim for a sheaf E or its shift $E[1]$ in $\mathcal{A}(b)$.

Let $E \in \mathcal{A}(b)$ be a coherent sheaf and $E' \subset E$ be a subobject of E in $\mathcal{A}(b)$, then E' is also a sheaf. Write $v_H(E) = (r, c, h)$. Let $\{E'_i\}_{i=1}^{i=n}$ be the HN factors of E' with respect to μ_H-stability. By definition of the heart $\mathcal{A}(b)$, these factors are also in the heart and

$$0 \leq \mathrm{Im}[Z_{(b,a)}(E'_i)] \leq \mathrm{Im}[Z_{(b,a)}(E')] \leq \mathrm{Im}[Z_{(b,a)}(E)] = c - rb.$$

Assume the factor E'_i has rank $r'_i \neq 0$, then the Bogomolov inequality for μ_H-semistable sheaf E_i' implies that

$$\mathrm{Re}[Z_{(b,a)}(E_i')] = \frac{-1}{2r_i'}\big(\mathrm{Im}[Z_{(b,a)}(E_i')]\big)^2 + \frac{\overline{\Delta}_H(E_i')}{2r_i'} + \frac{r_i'a^2}{2} \geq \frac{-1}{2r_i'}\big(\mathrm{Im}[Z_{(b,a)}(E_i')]\big)^2.$$

There is a short exact sequence in $\mathcal{A}(b)$ of coherent sheaves,

$$0 \to T' \to E' \to E'/T' \to 0$$

where T' is the torsion part of E'. Then

$$\mathrm{Re}[Z_{(b,a)}(E'/T')] = \sum_{i=k}^{n} \mathrm{Re}[Z_{(b,a)}(E_i')] \geq -\sum_{i=1}^{n} \big(\mathrm{Im}[Z_{(b,a)}(E_i')]\big)^2 \geq -\big(\mathrm{Im}[Z_{(b,a)}(E')]\big)^2,$$

where $k = 1$, if E' is a torsion free sheaf and $k = 2$, if $T' \neq 0$. Therefore,

$$\mathrm{Re}[Z_{(b,a)}(E'/T')] \geq -\big(\mathrm{Im}[Z_{(b,a)}(E)]\big)^2. \tag{2.6}$$

Consider the composition of injections $f\colon T' \hookrightarrow E' \hookrightarrow E$, then there is a short exact sequence in $\mathcal{A}(b)$

$$0 \to T' \xrightarrow{f} E \to E'' \to 0.$$

Taking cohomology implies that we have a long exact sequence of coherent sheaves

$$0 \to H^{-1}(E'') \to T' \to E \to H^0(E'') \to 0.$$

By definition $H^{-1}(E'')$ is a torsion free sheaf, so $H^{-1}(E'') = 0$ and T' is a subsheaf of E and so a subsheaf of its torsion part $T(E)$. Assume $v_H(T') = (0, c', h')$, then

$$0 \leq \mathrm{Im}[Z_{(b,a)}(T')] = c' \leq c - br \quad \text{and} \quad \mathrm{Re}[Z_{(b,a)}(T')] = bc' - h'.$$

The existence of a HN filtration for $T(E)$ with respect to Gieseker stability implies that h' is bounded from above, so the claim follows.

Now consider an object $E[1] \in \mathcal{A}(b)$ where E is a torsion free sheaf. To prove the claim, it is enough to show that for any quotient $E[1] \twoheadrightarrow E''$, the real part $\mathrm{Re}[Z_{(b,a)}(E'')]$ is bounded from above. Taking cohomology implies that $E'' = F[1]$ for a torsion free sheaf F. Let $\{F_i\}_{i=1}^{i=m}$ be the HN factors of F with respect to μ_H-stability. If the factor F_i is of rank r_i, then

$$\mathrm{Re}[Z_{(b,a)}(F_i[1])] = \frac{1}{2r_i}\big(\mathrm{Im}[Z_{(b,a)}(F_i[1])]\big)^2 - \frac{\overline{\Delta}_H(F_i)}{2r_i} - \frac{r_i'a^2}{2} \leq \frac{1}{2r_i'}\big(\mathrm{Im}[Z_{(b,a)}(F_i[1])]\big)^2.$$

This implies

$$\mathrm{Re}[Z_{(b,a)}(E'')] \leq \sum_{i=1}^{m} \big(\mathrm{Im}[Z_{(b,a)}(F_i[1])]\big)^2 \leq \big(\mathrm{Im}[Z_{(b,a)}(E[1])]\big)^2, \tag{2.7}$$

and the claim follows. □

If the real number b changes in a bounded interval, the final bounds in inequalities (2.6) and (2.7) also lie in a bounded area. So, we have the following result.

Corollary 2.3.7. *Fix a real number $a \in \mathbb{R}^{>0}$ and a bounded interval $I \subset \mathbb{R}$. Suppose* *the object E is in the heart $\mathcal{A}(b)$ for all $b \in I$, then there is a compact subset $V \subset \mathbb{C}$ which satisfies the following: if F is a subobject of E in $\mathcal{A}(b_0)$ for some $b_0 \in I$ and $\phi_{(b_0,a)}(F) \geq \phi_{(b_0,a)}(E)$, then the complex number $Z_{(b_0,a)}(F)$ lies in V.*

Given an object $E \in \mathcal{A}(b)$, we say that E has a destabilising subobject of maximum phase with respect to $\nu_{(b,a)}$ if there is a subobject $E' \subset E$ such that for any other subobject $F \subset E$, we have $\phi_{(b,a)}(E') \geq \phi_{(b,a)}(F)$.

As a result of Lemma 2.3.6, one can show that any object $E \in \mathcal{A}(b)$ has a destabilising subobject of maximum phase with respect to $\nu_{(b,a)}$, if $b \in \mathbb{Q}$. Indeed, Lemma 2.3.3 implies that for any subobject $F \subset E$ with bigger phase, there exists a subobject $F' \subset F$ which is $\nu_{(b,a)}$-semistable and $\phi_{(b,a)}(F') \geq \phi_{(b,a)}(F)$. To find a destabilising subobject of E of maximum phase, it is therefore enough to consider $\nu_{(b,a)}$-semistable subobjects $F' \subset E$, which have non-negative discriminant by Lemma 2.3.5. Lemma 2.3.6 also implies that the complex number $Z_{(b,a)}(F')$ lies in a compact subset of $\mathbb{C}$. Thus, there are only finitely many possible classes in Λ for the subobject F', which implies that there is a subobject $E' \subset E$ with the maximum phase.

Lemma 2.3.8. *Given a pair $(b,a) \in \mathbb{R} \times \mathbb{R}^{>0}$. Suppose any object $E \in \mathcal{A}(b)$ has a destabilising subobject of maximum phase with respect to $\nu_{(b,a)}$. If $\overline{\Delta}_H(F) \geq 0$ for any $\nu_{(b,a)}$-semistable object, then $\nu_{(b,a)} = (Z_{(b,a)}, \mathcal{A}(b))$ is a weak stability condition on $\mathcal{D}(X)$.*

Proof. It is enough to show that a Harder-Narasimhan filtration exists for any object $E \in \mathcal{A}(b)$. If E is stable, then $0 \subset E$ is the HN filtration. Otherwise, let E_1 be a $\nu_{(b,a)}$-semistable subobject of E of maximum phase. If E/E_1 is stable, then a HN filtration of E is

$$E_1 \hookrightarrow E \twoheadrightarrow E/E_1.$$

Otherwise, let F_2 be a subobject of E/E_1 with the maximum phase with respect to $\nu_{(b,a)}$. There is a short exact sequence in $\mathcal{A}(b)$

$$F_2 \hookrightarrow E/E_1 \twoheadrightarrow G.$$

Thus we have the surjection $f\colon E \twoheadrightarrow E/E_1 \twoheadrightarrow G$ in $\mathcal{A}(b)$. Let $E_2 = \ker(f)$, then $F_2 = E_2/E_1$ and we have the sequence of subobject $0 \subset E_1 \subset E_2 \subset E$. By assumption, $\phi_{(b,a)}(E_1) \geq \phi_{(b,a)}(E_2)$, so

$$\phi_{(b,a)}(E_1) \geq \phi_{(b,a)}(E_2/E_1). \tag{2.8}$$

Continuing this process gives the sequence

$$0 = E_0 \subset E_1 \subset E_2 \subset .... \subset E_i \subset E_{i+1} \subset .... \subset E \tag{2.9}$$

such that E_{i+1}/E_i is $\nu_{(b,a)}$-semistable and $\phi_{(b,a)}(E_{i+1}/E_i) \geq \phi_{(b,a)}(E_{i+2}/E_{i+1})$. Lemma 2.3.6 implies that there is a bounded subset $V \subset \mathbb{C}$ such that the complex number $Z_{(b,a)}(E_{i+1}/E_i)$ lies in V for all i. By assumption, the factors E_{i+1}/E_i have non-negative discriminant. Therefore, there are only finitely many classes in Λ for $v_H\big(E_{i+1}/E_i\big)$, which implies the chain (2.9) terminates. Finally, if for some i, we have $\phi_{(b,a)}(E_i/E_{i-1}) = \phi_{(b,a)}(E_{i+1}/E_i)$, then we remove the term E_i from the chain (2.9), so we get a sequence with strictly decreasing phases. Note that since E_i/E_{i-1} and E_{i+1}/E_i are $\nu_{(b,a)}$-semistable of the same phase, their extension E_{i+1}/E_{i-1}

$$E_i/E_{i-1} \hookrightarrow E_{i+1}/E_{i-1} \twoheadrightarrow E_{i+1}/E_i$$

is also $\nu_{(b,a)}$-semistable. $\square$

Lemma 2.3.5 and 2.3.8 imply that the pair $\nu_{(b,a)} = \big(Z_{(b,a)}, \mathcal{A}(b)\big)$ is a weak stability condition on $\mathcal{D}(X)$ if the pair $(b,a) \in \mathbb{Q} \times \mathbb{R}^{>0}$. The next step is to figure out how these weak stability conditions change when the numbers b and a vary.

Lemma 2.3.9. *We have the torsion pair $\big(\mathcal{A}(b') \cap \mathcal{A}(b)[1]\ ,\ \mathcal{A}(b) \cap \mathcal{A}(b')\big)$ in $\mathcal{A}(b')$ if $b' \geq b$. For any object $E \in \mathcal{A}(b') \cap \mathcal{A}(b)$, if we have an injective map*

$$E' \hookrightarrow E$$

in $\mathcal{A}(b')$, then the object $E' \in \mathcal{A}(b)$.

Proof. The first part is clear by definition. For the second part, the object E' is an extension

$$T \hookrightarrow E' \twoheadrightarrow F$$

for $T \in \mathcal{A}(b') \cap \mathcal{A}(b)[1]$ and $F \in \mathcal{A}(b') \cap \mathcal{A}(b)$. If $T \neq 0$, then consider the composition of injections $T \hookrightarrow E' \hookrightarrow E$ in $\mathcal{A}(b')$, so we have a non-zero map from $T \in \mathcal{A}(b)[1]$ to $E \in \mathcal{A}(b)$, a contradiction. Therefore, $T = 0$ and $E' = F \in \mathcal{A}(b)$. $\square$

Fix a pair $(b,a) \in \mathbb{R} \times \mathbb{R}^{>0}$. Consider the path of weak stability functions given by

$$\gamma \colon t \in [0,1] \to \nu_{(b+t,a)} = (Z_{(b+t,a)}, \mathcal{A}(b+t)).$$

Given an object $E \in \mathcal{A}(b+t_0) \cap \mathcal{A}(b)$ for some $t_0 \leq 1$. Assume there is a short exact sequence

$$E' \hookrightarrow E \twoheadrightarrow E''$$

in $\mathcal{A}(b+t_0)$ but not in $\mathcal{A}(b)$. Lemma 2.3.9 implies that $E' \in \mathcal{A}(b)$. Therefore, $E'' \notin \mathcal{A}(b)$ and we have the short exact sequence

$$G[1] \hookrightarrow E'' \twoheadrightarrow F \tag{2.10}$$

in $\mathcal{A}(b+t_0)$ for two objects $F, G \in \mathcal{A}(b)$. Since $G[1] \in \mathcal{A}(b)[1] \cap \mathcal{A}(b+t_0)$, we have $H^{-1}(G) = 0$ and so G is a sheaf. As the real number t varies, we have $G[1] \in \mathcal{A}(b+t)$, and thus $E'' \in \mathcal{A}(b+t)$ if and only if $t \geq \mu_H^+(G) - b =: t^*$.

Lemma 2.3.10. *Fix an object $E \in \mathcal{A}(b) \cap \mathcal{A}(b+t^*)$. Assume there is a quotient $E \twoheadrightarrow E''$ in the heart $\mathcal{A}(b+t^*)$ but not in $\mathcal{A}(b+t)$ for $t \in [0, t^*)$. Suppose $\phi_{(b+t^*,a)}(E'') < 1$. Then there is a quotient $E \twoheadrightarrow E_1''$ in the heart $\mathcal{A}(b+t)$ for $t \in (t^* - \epsilon, t^*]$ where $\epsilon > 0$ is sufficiently small, such that*

$$\phi_{(b+t^*,a)}(E_1'') < \phi_{(b+t^*,a)}(E'').$$

Proof. There is a short exact sequence

$$G[1] \hookrightarrow E'' \twoheadrightarrow F \tag{2.11}$$

in $\mathcal{A}(b+t^*)$ for $F, G \in \mathcal{A}(b)$. Let G_1 be the subsheaf of G with maximum μ_H-slope and G/G_1 be the quotient sheaf, so there is a short exact sequence

$$G_1[1] \hookrightarrow G[1] \twoheadrightarrow G/G_1[1]$$

in $\mathcal{A}(b+t^*)$. If G is μ_H-semistable, we assume $G_1 = G$. The object $G_1[1]$ is $\nu_{(b+t^*,a)}$-semistable of phase one. Consider the composition of injections $f\colon G_1[1] \hookrightarrow G[1] \hookrightarrow E''$. Then there is a short exact sequence

$$G_1[1] \hookrightarrow E'' \twoheadrightarrow E_1''$$

in $\mathcal{A}(b+t^*)$ where $E_1'' = \operatorname{coker}(f)$. Since $G_1[1]$ is of phase one and phase of E'' is less than one, we have $\phi_{(b+t^*,a)}(E_1'') < \phi_{(b+t^*,a)}(E'')$. There is a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G_1[1] \arrow[r] \arrow[d, "\cong"] & G[1] \arrow[r] \arrow[d] & G/G_1[1] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_1[1] \arrow[r, "f"] & E'' \arrow[r] \arrow[d] & E_1'' \arrow[r] \arrow[d] & 0 \\
 & & F \arrow[r, "\cong"] \arrow[d] & F \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

If $G/G_1 = 0$, then G_1 is μ_H-semistable and $E_1'' = F$ which is in the heart $\mathcal{A}(b)$. Otherwise, $E_1'' \in \mathcal{A}(b+t)$ if $t \geq \mu_H^+(G/G_1) - b$ which proves the claim. $\square$

Lemma 2.3.11. *Fix an object $E \in \mathcal{A}(b) \cap \mathcal{A}(b+t^*)$. Assume there is a subobject $E' \hookrightarrow E$ in $\mathcal{A}(b+t^*)$ but E' is not a subobject of E in $\mathcal{A}(b+t)$ for $t \in [0, t^*)$. Then there is a subobject $E_1' \hookrightarrow E$ in the heart $\mathcal{A}(b+t)$ for $t \in (t^* - \epsilon, t^*]$ where $\epsilon > 0$ is sufficiently small, such that*

$$\phi_{(b+t^*,a)}(E_1') > \phi_{(b+t^*,a)}(E').$$

Proof. In the above notations, consider the composition of surjections $g\colon E \twoheadrightarrow E'' \twoheadrightarrow E_1''$ in $\mathcal{A}(b+t^*)$. Then there is a short exact sequence

$$\ker(g) \hookrightarrow E \twoheadrightarrow E_1''$$

in $\mathcal{A}(b+t)$ for $t \in (t^* - \epsilon, t^*]$. Let $E_1' \coloneqq \ker(g)$, then there is a short exact sequence

$$E' \hookrightarrow E_1' \twoheadrightarrow G_1[1]. \tag{2.12}$$

in $\mathcal{A}(b+t^*)$. Since $\phi_{(b+t^*,a)}(G_1[1]) = 1$, we have $\phi_{(b+t^*,a)}(E_1') \geq \phi_{(b+t^*,a)}(E')$. If we have equality, then $\phi_{(b+t^*,a)}(E') = \phi_{(b+t^*,a)}(E') = 1$. Since the objects E' and E_1' are in the heart $\mathcal{A}(b+t)$ for $t \in (t^* - \epsilon, t^*]$, they must be sheaves supported in co-dimension at least two, which is impossible by the short exact sequence (2.12). $\square$

Remark 2.3.12. If an object E is in $\mathcal{A}(b+t)$ for $t_1 < t < t_2$ and $\overline{\Delta}_H(E) \geq 0$, then the equality (2.4) implies that the phase $\phi_{(b+t,a)}(E)$ is decreasing with respect to t.

Lemma 2.3.13. *Fix a pair $(b,a) \in \mathbb{Q} \times \mathbb{R}^{>0}$ and an object $F \in \mathcal{D}(X)$. The maximum phase $\phi^+_{(b+t,a)}(F)$ changes continuously with respect to $t \in [0,1] \cap \mathbb{Q}$.*

Proof. Define $j := \min\{i\colon \mathcal{H}^i_{\mathcal{A}(b)}(F) \neq 0\}$ and $E := \mathcal{H}^j_{\mathcal{A}(b)}(F) \in \mathcal{A}(b)$. Let $t_0 = \mu^-_H(H^0(E)) - b$, then for any $t \in [0, t_0)$, the object $E \in \mathcal{A}(b+t)$. Moreover,

$$\min\{i\colon \mathcal{H}^i_{\mathcal{A}(b+t)}(F) \neq 0\} = j \quad \text{and} \quad \mathcal{H}^j_{\mathcal{A}(b+t)}(F) = E.$$

If $t \in [0, t_0) \cap \mathbb{Q}$, then

$$\phi^+_{(b+t,a)}(F) = \phi^+_{(b+t,a)}(E) - j.$$

We first show that the function $\phi^+_{(b+t,a)}(E)$ is continuous on the interval $[0, t_0) \cap \mathbb{Q}$. Assume we have a subobject $E' \subset E$ in $\mathcal{A}(b+t)$ with $\phi_{(b+t,a)}(E') > \phi_{(b+t,a)}(E)$ for some $t \in [0, t_0) \cap \mathbb{Q}$. Then, Corollary 2.3.7 implies that the complex number $Z_{(b+t,a)}(E')$ lies in a bounded area. By Lemma 2.3.5, there are only finitely many possible classes $v \in \Lambda$ for the subobject E' that gives $\phi^+_{(b+t,a)}(E)$ for some $t \in [0, t_0) \cap \mathbb{Q}$. There is a sequence of real numbers $0 = k_1 < k_2 < k_3 < \dots < k_{n-1} < k_n = t_0$ and vectors $v_i \in \Lambda$ for $0 \leq i < n$ such that

$$\phi^+_{(b+t,a)}(E) = \phi_{(b+t,a)}(v_i) \quad \text{for } t \in (k_i, k_{i+1}) \cap \mathbb{Q}. \tag{2.13}$$

Thus it is enough to show that $\phi_{(b+k_i,a)}(v_{i-1}) = \phi_{(b+k_i,a)}(v_i)$ for $1 < i < n$ and $\phi^+_{(b,a)}(E) = \phi_{(b,a)}(v_1)$. Assume E_i is a subobject of E in $\mathcal{A}(b+t_i)$ for some $t_i \in (k_i, k_{i+1})$ and $v_H(E_i) = v_i$, so it is $\nu_{(b+t_i,a)}$-semistable and there is a short exact sequence

$$E_i \hookrightarrow E \twoheadrightarrow E''_i$$

in $\mathcal{A}(b+t_i)$. We claim that the object E_i is a subobject of E in $\mathcal{A}(b+k_i)$. Lemma 2.3.9 implies that $E_i \in \mathcal{A}(b+k_i)$. If the object $E''_i \in \mathcal{A}(b+k_i)$, then the claim holds. Otherwise, we have

$$b + k_i < t^* := \mu^+_H\big(H^{-1}(E''_{i+1})\big).$$

Lemma 2.3.11 implies that there exists a subobject $E'_i \subset E$ in $\mathcal{A}(b+t^*)$ with

$$\phi_{(b+t^*,a)}(E'_i) > \phi_{(b+t^*,a)}(E_i)$$

which is a contradiction by assumption (2.13). Applying a dual argument implies that there is a subobject $E_{i-1} \subset E$ in $\mathcal{A}(b+t_{i-1})$ for $t_{i-1} \in (k_{i-1}, k_i)$ such that $v_H(E_{i-1}) = v_{i-1}$ and $E_{i-1} \subset E$ in $\mathcal{A}(b+k_i)$. We claim that

$$\phi_{(b+k_i,a)}(E_{i-1}) = \phi_{(b+k_i,a)}(E_i) \quad \text{for } 1 < i < n.$$

By definition of the heart, the object E_{i-1} is a subobject of E in $\mathcal{A}(b+t)$ for $t \in [k_i, k_i+\epsilon)$ where ϵ is sufficiently small, thus

$$\phi_{(b+k_i,a)}(E_{i-1}) \leq \phi_{(b+k_i,a)}(E_i).$$

If the object E_i is a subobject of E in $\mathcal{A}(b+t)$ for $t \in (k_i - \epsilon, k_i]$, then

$$\phi_{(b+k_i,a)}(E_i) \leq \phi_{(b+k_i,a)}(E_{i-1})$$

and the claim holds. Otherwise, Lemma 2.3.11 implies that there is a subobject $E'_i \in \mathcal{A}(b+t)$ for $t \in (k_i - \epsilon, k_i]$ such that $\phi_{(b+k_i,a)}(E'_i) > \phi_{(b+k_i,a)}(E_i)$ which is a contradiction by assumption (2.13). Applying a similar argument implies that $\phi^+_{(b,a)}(E) = \phi_{(b,a)}(v_1)$ which implies the function $\phi^+_{(b+t,a)}$ is continuous on $[0, t_0) \cap \mathbb{Q}$.

If $\mathcal{H}^j_{\mathcal{A}(b+t_0)}(F) \neq 0$, we have the injection

$$\mathcal{H}^j_{\mathcal{A}(b+t_0)}(F) \hookrightarrow E \twoheadrightarrow E'$$

in $\mathcal{A}(b+t)$ where $t \in (t_0 - \epsilon, t_0)$ and ϵ is sufficiently small. Therefore,

$$\phi^+_{(b+t,a)}(E) = \phi^+_{(b+t,a)}(\mathcal{H}^j_{\mathcal{A}(b+t_0)}(F))$$

which gives continuity of $\phi^+_{(b+t,a)}(E)$ at the time $t = t_0$. If $\mathcal{H}^j_{\mathcal{A}(b+t_0)}(F) = 0$, then

$$\lim_{t \to t_0} \phi^+_{(b+t,a)}(E) = 0 \quad \Rightarrow \quad \lim_{t \to t_0} \phi^+_{(b+t,a)}(F) = -j.$$

Lemma 2.3.14 shows that E is a μ_H-semistable sheaf. Therefore, $E[1] \in \mathcal{A}(b+t_0)$ and it is semistable of phase one. At the time $t = t_0$, we have

$$\max\{i \colon \mathcal{H}^i_{\mathcal{A}(b+t_0)}(F) \neq 0\} = j+1 \quad \Rightarrow \quad \phi^+_{(b+t_0,a)}(F) = \phi^+(\mathcal{H}^{j+1}_{\mathcal{A}(b+t_0)}(E)) - j - 1.$$

There is an injection

$$E[1] \hookrightarrow \mathcal{H}^{j+1}_{\mathcal{A}(b+t_0)}(E)$$

in $\mathcal{A}(b+t_0)$, so $\phi^+_{(b+t_0,a)}(E[1]) = \phi^+(\mathcal{H}^{j+1}_{\mathcal{A}(b+t_0)}(E)) = 1$, which implies $\phi^+_{(b+t_0,a)}(F) = -j$. Therefore, $\phi^+_{(b+t,a)}(F)$ is continuous at the time $t = t_0$. Continuing this argument for $t \geq t_0$ implies that $\phi^+_{(b+t,a)}(F)$ is continuous on the entire interval. $\square$

Lemma 2.3.14. *Given real numbers $k' < k$. Suppose the object E is in the heart $\mathcal{A}(b+t)$ for any $t \in (k', k)$. If*

$$\lim_{t \to k} \phi^+_{(b+t,a)}(E) = 0,$$

then E is a μ_H-semistable sheaf.

Proof. If an object $F \in \mathcal{A}(b+t)$ has negative rank, then the function $\mathrm{Im}[Z_{(b+t,a)}(F)]$ is increasing with respect to t, so its phase cannot go to zero. Since the phase $\phi^+_{(b+t)}(E)$ is going to zero, the phase of its subobject $H^{-1}(E)$ is also going to zero. Therefore, $H^{-1}(E) = 0$. Moreover,

$$\lim_{t \to k} \mathrm{Im}[Z_{(b+t,a)}(E)] = 0,$$

which gives $k+b = \mu_H(E)$. Thus the result follows by definition of the heart $\mathcal{A}(b+t)$. $\square$

Proposition 2.3.15. *The pair $\nu_{(b,a)} = \big(Z_{(b,a)}, \mathcal{A}(b)\big)$ is a weak stability condition for any pair $(b,a) \in \mathbb{R} \times \mathbb{R}^{>0}$.*

Proof. Given an object $E \in \mathcal{A}(b)$ where $b \notin Q$. The argument of Lemma 2.3.13 implies that there are real numbers $b_1 < b_2$ and an object $E' \in \mathcal{D}(X)$ such that $b \in [b_1, b_2]$ and $\phi^+_{(b',a)}(E) = \phi_{(b',a)}(E')$ for any $b' \in [b_1, b_2] \cap \mathbb{Q}$. We claim that E' is a destabilising subobject of E of maximum phase. Assume for a contradiction that there exists a subobject $E'' \subset E$ in $\mathcal{A}(b)$ with

$$\phi_{(b,a)}(E'') > \phi_{(b,a)}(E').$$

Since $b \in \mathbb{R} \setminus \mathbb{Q}$, the object E'' is a subobject of E in $\mathcal{A}(b+t)$ for $t \in (-\epsilon, +\epsilon)$ where $\epsilon > 0$ is sufficiently small. This leads to a contradiction.

If E is $\nu_{(b,a)}$-stable, then it is $\nu_{(b',a)}$-semistable for any $b' \in [b_1, b_2] \cap \mathbb{Q}$. Therefore, Lemma 2.3.5 implies that $\overline{\Delta}_H(E) \geq 0$. The result follows by Lemma 2.3.8. $\square$

Proof of Theorem 2.3.1. A similar argument as in Lemma 2.3.13 and applying Lemma 2.3.10 imply that for any object $F \in \mathcal{D}(X)$, the phase $\phi^-_{(b,a)}(F)$ changes continuously with respect to b. We claim that for any pair $(b,a) \in \mathbb{R} \times \mathbb{R}^{>0}$,

$$\lim_{t \to 0} \sup_{F \in \mathcal{D}(X)} \left| \phi^{\pm}_{(b+t,a)}(F) - \phi^{\pm}_{(b,a)}(F) \right| = 0. \tag{2.14}$$

We can divide the interval $[0,t] = \bigcup_i [k_i, k_{i+1}]$ such that for any i, there exists an object $E_i \in \mathcal{D}(X)$ with $\phi^+_{(b+t,a)}(F) = \phi_{(b+t,a)}(E_i)$ for $t \in [k_i, k_{i+1}]$. Therefore,

$$\left| \phi^+_{(b+t,a)}(F) - \phi^+_{(b,a)}(F) \right| \leq \sum_i \left| \phi^+_{(b+k_{i+1},a)}(F) - \phi^+_{(b+k_i,a)}(F) \right| =$$

$$\sum_i \left| \phi_{(b+k_{i+1},a)}(E_i) - \phi_{(b+k_i,a)}(E_i) \right|.$$

Definition of the phase function and equality (2.4) implies that for any object E with $\overline{\Delta}_H(E) \geq 0$, we have $P \coloneqq \overline{\Delta}_H(E)/2r^2 + a^2/2 \geq a^2/2$ and

$$\left| \frac{d\phi_{(b,a)}(E)}{db} \right| = \frac{x^2/2 + P}{\pi\big((x^2/2 - P)^2 + x^2\big)},$$

where $x = b - c/r$. If $(x^2/2 - P)^2 \geq (P/2)^2$, then

$$\left| \frac{d\phi_{(b,a)}(E)}{db} \right| \leq \frac{2x^2 + 4P}{\pi(4x^2 + P^2)} \leq \max\left\{ \frac{8}{a^2\pi}, \frac{1}{2\pi} \right\}.$$

If $(x^2/2 - P)^2 \leq (P/2)^2$, i.e. $P \leq x^2 \leq 3P$, then

$$\left| \frac{d\phi_{(b,a)}(E)}{db} \right| \leq \frac{x^2/2 + P}{\pi x^2} \leq \frac{5}{2\pi}.$$

Let $M \coloneqq \max\left\{ \frac{8}{a^2\pi}, \frac{5}{2\pi} \right\}$, then for any i,

$$\left| \frac{d\phi_{(b+t,a)}(E_i)}{dt} \right| \leq M \quad \Rightarrow \quad \left| \phi_{(b+k_{i+1},a)}(E_i) - \phi_{(b+k_i,a)}(E_i) \right| \leq (k_{i+1} - k_i)M.$$

This finally gives

$$\left| \phi^+_{(b+t,a)}(F) - \phi^+_{(b,a)}(F) \right| \leq \sum_i (k_{i+1} - k_i)M = Mt.$$

The upper bound does not depend on the object F, so (2.14) is satisfied for ϕ^+ and the same argument proves the claim for ϕ^-.

On the other hand, a similar argument as in Lemma 2.3.13 implies that the phase $\phi^{\pm}_{(b,a)}(F)$ is continuous with respect to a for any object $F \in \mathcal{D}(X)$. By equality (2.5), we have

$$\frac{d\phi_{(b,a)}(F)}{da} = \frac{ax}{\pi\big((x^2/2 - P)^2 + x^2\big)}.$$

A similar argument as above shows that

$$\left|\frac{d\phi_{(b,a+t)}(F)}{dt}\right| \leq \max\left\{\frac{4}{a^2}, 2\right\},$$

so the result follows. □

The two dimensional family of weak stability conditions satisfies well-behaved wall-crossing.

Proposition 2.3.16 ([Bri08, BMS16]). *Fix an object $E \in \mathcal{D}(X)$ with $v_H(E) \neq 0$. There exists a wall and chamber structure given by a locally finite set of walls in $\mathbb{R}\times\mathbb{R}^{>0}$ such that $\nu_{(b,a)}$-(semi)stability is unchanged as the pair (b,a) varies within a chamber. Moreover, a weak stability condition $\nu_{(b,a)}$ is on a wall $\mathcal{W}_E$ corresponding to the object E if and only if the following conditions hold:*

(a) Some shift $E[k]$ is in the heart $\mathcal{A}(b)$ and it is strictly $\nu_{(b,a)}$-semistable.
(b) There is a subobject $F \subset E[k]$ in $\mathcal{A}(b)$ such that $\phi_{(b,a)}(F) = \phi_{(b,a)}(E[k])$ and $v_H(F) \neq k v_H(E)$ for any $k \in \mathbb{R}\setminus\{0\}$.

Chapter 3

An Effective Restriction Theorem for Stable Vector Bundles

3.1 Introduction

The question of whether the stability of a vector bundle with respect to an ample divisor is preserved under the restriction to a hypersurface has been studied since the 1980s. There are several approaches to this question, see [HL10, Chapter 7] for a survey. Mehta and Ramanathan proved that the restriction of a μ-semistable sheaf to a general hypersurface of sufficiently high degree is semistable [MR84, MR82]; However, this bound is not effective. In the characteristic zero case, Flenner proved an effective restriction theorem which provides a bound on the degree of the hypersurface depending only on the rank of the sheaf and the degree of the variety [Fle84].

Using one of these restriction theorems, one can prove the Bogomolov inequality over a characteristic zero base field, which states the discriminant of any μ-semistable torsion free sheaf is non-negative [HL10, Theorem 7.3.1]. This result gives Bogomolov's restriction theorem, which is a stronger effective restriction theorem. Langer generalised this result to varieties over any base field. In [Lan04, Lan10], he proved several effective restriction theorems for μ-semistable torsion free sheaves on a smooth projective variety over an algebraically closed field k of characteristic p. The following is a variant of one of his results that we will reprove in this chapter via wall-crossing.

Theorem 3.1.1 ([Lan04])**.** *Let X be a smooth projective complex variety of dimension $n \geq 2$. Let H be an ample divisor on X and let E be a μ_H-stable vector bundle of rank $rk \geq 2$ on X. Take an integer l such that*

$$l > \frac{rk-1}{rk}\,\frac{\overline{\Delta}_H(E)}{H^n} + \frac{1}{H^n rk(rk-1)},$$

where $\overline{\Delta}_H(E) = \left(H^{n-1}ch_1(E)\right)^2 - 2rkH^n ch_2(E)H^{n-2}$. Then for any divisor $C \in |lH|$, the restriction $E|_C$ is $\mu_{H|_C}$-stable

As we discussed in Chapter 2, the Bogomolov inequality implies that we have weak stability conditions on any smooth projective variety. Wall-crossing with respect to these stability conditions provides a direct way to prove slope stability of the restricted bundle. Indeed, we only need to find a weak stability condition ν such that both E and $E(-lH)[1]$ are ν-stable of the same phase. Then push-forward of the restricted bundle $i_*\big(E|_C\big)$ for a divisor $C \in |lH|$ with embedding $i\colon C \hookrightarrow X$, fits into the short exact sequence

$$E \hookrightarrow i_*\big(E|_C\big) \twoheadrightarrow E(-lH)[1].$$

Therefore, $i_*E|_C$ is ν-semistable. By changing ν in the right direction, we can get its strict stability. Then a general argument immediately implies that $E|_C$ is $\mu_{H|_C}$-stable.

This strategy first appeared in [Fey16] for the case that X is a K3 surface. It has been later generalised in [Kop18] to any surface. The author showed that this method can be used to investigate the stability of the restriction of a vector bundle to any divisor on the surface not just multiples of H. In this paper, he considered semicircle walls and the proof is different from the one that we present here.

This method also gives a stronger result than Langer's Theorem 3.1.1 in special cases.

Theorem 3.1.2. *Let X be a smooth projective complex variety of dimension $n \geq 2$. Let H be an ample divisor on X such that for any divisor H', we have $H^n | H'.H^{n-1}$. Assume E is a μ_H-stable vector bundle of rank $rk \geq 2$ and the integer l satisfies*

$$l > \frac{rk-1}{rk}\,\frac{\overline{\Delta}_H(E)}{(H^n)^2} + \frac{1}{rk(rk-1)}.$$

Then for any divisor $C \in |lH|$, the restriction $E|_C$ is $\mu_{H|_C}$-stable.

As a consequence, we can prove the stability of the restriction of Lazarsfeld-Mukai bundles associated to line bundles on curves in K3 surfaces. This gives interesting counterexamples to Mercat's conjecture [Fey16].

3.2 Two-dimensional slice of weak stability conditions

In this section, we define a projection map and show that the walls can be seen as line segments in the projected space.

Assume X is a smooth projective complex variety of dimension n. Let H be an ample line bundle on X. As shown in Chapter 2, there is a two-dimensional slice of weak stability conditions given by $\nu_{(b,a)} = \big(Z_{(b,a)}, \mathcal{A}(b)\big)$ for $(b,a) \in \mathbb{R} \times \mathbb{R}^{>0}$. We defined the lattice Λ as the image of the homomorphism $v_H \colon K(X) \to \mathbb{Z} \oplus \mathbb{Z} \oplus \dfrac{\mathbb{Z}}{2}$ given by

$$v_H(E) = \big(\mathrm{ch}_0(E)H^n, \mathrm{ch}_1(E)H^{n-1}, \mathrm{ch}_2(E)H^{n-2}\big).$$

To simplify drawing figures, we always think of the projection of the lattice

$$pr \colon \Lambda \setminus \{h=0\} \to \mathbb{R}^2 \ \ , \ \ pr(r,c,h) = \left(\frac{c}{h}, \frac{r}{h}\right).$$

Recall that the stability function $Z_{(b,a)}$ for $(b,a) \in \mathbb{R} \times \mathbb{R}^{>0}$ is defined as

$$Z_{(b,a)} \colon \Lambda \to \mathbb{C} \ \ , \ \ Z_{(b,a)}(r,c,h) = \left\langle (r,c,h), \left(1, b, \frac{b^2-a^2}{2}\right)\right\rangle + i\big\langle (r,c,h), (0,1,b)\big\rangle.$$

There is a homeomorphism

$$k \colon \mathbb{R} \times \mathbb{R}^{>0} \to U = \{(x,y) \in \mathbb{R}^2 \colon y > \frac{x^2}{2}\} \ \ , \ \ k(b,a) = \left(\frac{2b}{b^2+a^2}, \frac{2}{b^2+a^2}\right),$$

with the inverse

$$k^{-1}\colon U \to \mathbb{R}\times\mathbb{R}^{>0} \quad , \quad k^{-1}(x,y) = \left(\frac{x}{y}\, , \, \sqrt{\frac{2y-x^2}{y^2}}\right).$$

Therefore, the two-dimensional family of weak stability conditions $\{\nu_{(b,a)}\colon (b,a)\in\mathbb{R}\times\mathbb{R}^{>0}\}$ is also continuously parametrised by the space U. Given a point $p\in U$, there exists a unique pair $(b,a)\in\mathbb{R}\times\mathbb{R}^{>0}$ such that $p=k(b,a)$, and the corresponding weak stability condition is $\nu_{(b,a)}$. Note that the point $k(b,a)$ is on a line with the equation

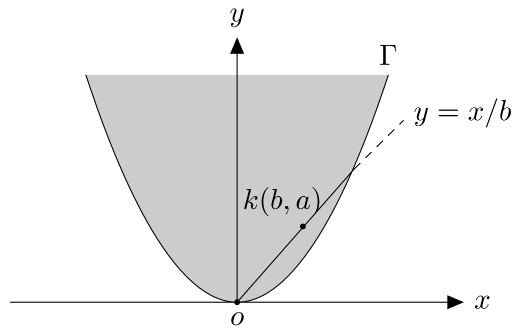

Figure 3.1: The space U

$y = x/b$, so when the point $k(b,a)$ moves along a line passing through the origin of $\mathbb{R}^2$, b is fixed and only a is changing. This means in the corresponding weak stability condition, the heart $\mathcal{A}(b)$ is fixed and only the stability function is changing. When $a\to+\infty$, the point $k(b,a)$ gets closer to the origin and when $a\to 0$, the point $k(b,a)$ moves towards the parabola Γ with equation $y = x^2/2$.

For simplicity, we use the same xy-plane for the space U and projection of the lattice $pr(\Lambda)$. Note that the kernel of stability function $Z_{(b,a)}$ is on the line spanned by the vector $\big(1, b, (b^2+a^2)/2\big)$ in $\Lambda_{\mathbb{R}}$. So the point $k(b,a)$ is indeed the projection $pr(\ker Z_{(b,a)})$.

Hence the open subset U is the projection of the negative cone in $\Lambda_{\mathbb{R}}$ with respect to the bilinear form $\langle -,-\rangle$. Lemma 2.3.5 and Proposition 2.3.15 imply that the Bogomolov inequality is satisfied for a $\nu_{(b,a)}$-semistable object $E\in\mathcal{D}(X)$, so $Q(v_H(E)) = \langle v_H(E), v_H(E)\rangle \geq 0$. Therefore, the point $pr(v_H(E))$ does not lie in the open subset U.

Lemma 3.2.1. *Suppose the objects E and E' are in the heart $\mathcal{A}(b_0)$ for some $b_0\in\mathbb{R}$. Assume $v_H(E) = (r,c,h)$ and $v_H(E') = (r',c',h')$ such that $h'\neq 0$. Suppose L is the line that passes through the points $pr(v_H(E'))$ and $pr(v_H(E))$ if $h\neq 0$; otherwise, it is the line that goes through the point $pr(v_H(E'))$ and has slope of r/c. The objects E and E' have the same phase with respect to the weak stability condition $\nu_{(b_0,a)}$ if and only if the point $k(b_0,a)$ is on the line L.*

Proof. The objects $E, E'\in\mathcal{A}(b_0)$ have the same phase with respect to the weak stability condition $\nu_{(b_0,a)}$ if and only if the line $\ker Z_{(b_0,a)}\subset\Lambda_{\mathbb{R}}$ is on the plane spanned by $v_H(E)$ and $v_H(E')$, i.e. its projection $k(b_0,a) = pr\big(\ker Z_{(b_0,a)}\big)$ is on the line L passing through the points $pr\big(v_H(E)\big)$ and $pr\big(v_H(E')\big)$ if $h\neq 0$.

If $h=0$, then any vector $v\in\ker Z_{(b_0,a)}$ can be written as a linear combination

$$v = x\big(r,c,0\big) + y\big(r',c',h'\big),$$

for some $x, y \in \mathbb{R}$. Thus, the point $pr(v) = k(b_0, a)$ is on the line L of slope r/c. $\square$

Proposition 2.3.16 implies that there is a locally finite set of the walls corresponding to any fixed object $E \in \mathcal{D}(X)$ in the upper half plane $\mathbb{R} \times \mathbb{R}^{>0}$. The next lemma describes the image of these walls under the isomorphism k.

Lemma 3.2.2. *Fix an object $E \in \mathcal{D}(X)$ with $v_H(E) = (r, c, h)$ such that r and c are not simultaneously zero. Any wall $\mathcal{W}_E \subset U$ corresponding to E is given by $L \cap U$ where L is a line that passes through the point $pr(v_H(E))$ if $h \neq 0$, or that has slope r/c if $h = 0$.*

Proof. Let $\nu_{(b,a)}$ be a weak stability condition on the wall $\mathcal{W}_E$. Proposition 2.3.16 implies that up to shift, the object E is in the heart $\mathcal{A}(b)$ and has a subobject E' of the same phase. Lemma 3.2.1 implies that the corresponding point $k(b, a)$ is on a line L as claimed. If $h = 0$, then $\mathrm{ch}_2(E') \neq 0$, otherwise E' cannot make a wall for E, i.e. there is no $(b, a) \in \mathbb{R} \times \mathbb{R}^{>0}$ such that E and E' have the same phase with respect to $Z_{(b,a)}$ if $v_H(E) \neq kv_H(E')$ for any $k \in \mathbb{R} \setminus \{0\}$.

As the point $k(b, a)$ varies on the line segment $L \cap U$, the phases of E and E' are fixed, so they remain inside the heart $\mathcal{A}(b)$. Note that the Bogomolov inequality implies that the points $pr(v_H(E))$ and $pr(v_H(E'))$ do not lie on the line segment $L \cap U$. Thus when $k(b, a)$ lies on the line segment $L \cap U$, we have $Z_{(b,a)}(E) \neq 0 \neq Z_{(b,a)}(E')$. $\square$

3.3 Slope stability of the restricted bundle

In this section, we describe sufficient conditions for a μ_H-stable vector bundle on a smooth projective variety that imply its restriction to a divisor remains stable.

As before, X is a smooth projective complex variety of dimension $n \geq 2$ and H is an ample line bundle on X. A similar argument as in [Bri08, Proposition 14.2] implies that there are some stability conditions in the large volume limit for which a μ_H-stable torsion free sheaf is stable.

Lemma 3.3.1. *Suppose E is a μ_H-stable torsion free sheaf with $v_H(E) = (r, c, h)$. Then E is $\nu_{(b,a)}$-stable if $b < c/r$ and $a \gg 0$.*

Proof. We may assume that $c > 0$, otherwise we replace E by the twist $E(kH)$ such that $\mathrm{ch}_1\big(E(kH)\big).H^{n-1} > 0$. Note that the object E is $\nu_{(b,a)}$-(semi)stable if and only if the twist $E(kH)$ is $\nu_{(b+k,a)}$-(semi)stable. The construction of the walls as described in Lemma 3.2.2, shows that it is enough to prove E is $\nu_{(0,a)}$-stable for $a \gg 0$, see Figure 3.2. Suppose

$$E' \hookrightarrow E \twoheadrightarrow E''$$

is a short exact sequence in $\mathcal{A}(0)$. Assume E' is semistable with respect to a weak stability condition $\nu_{(0,a)}$ and $v_H(E') = (r', c', h')$. Taking cohomology gives a long exact sequence of sheaves

$$0 \to H^{-1}(E'') \to E' \xrightarrow{f} E \to H^0(E'') \to 0.$$

By definition of the heart, $\mu_H(H^{-1}(E'')) \leq 0 < \mu_H(E')$, thus $\mu_H(E') < \mu_H(\mathrm{Im} f)$. The μ_H-stability of E implies

$$\mu_H(E') < \mu_H(E) \quad \Rightarrow \quad -\frac{r'}{c'} < -\frac{r}{c}. \tag{3.1}$$

In particular, it implies $c' > 0$. Moreover, $\mathrm{Im}[Z_{(0,a)}(E'')] = c - c' \geq 0$ which gives

$$-\frac{r'}{c'} \leq -\frac{r}{c} - \frac{1}{c^2}. \tag{3.2}$$

Since E' is $\nu_{(0,a)}$-semistable, the Bogomolov inequality implies that

$$0 \leq c'^2 - 2r'h' \quad \Rightarrow \quad \frac{h'}{c'} \leq \frac{c'}{2r'} < \frac{c}{2r}. \tag{3.3}$$

The last inequality comes from (3.1). The inequalities (3.2) and (3.3) show that

$$-\frac{\mathrm{Re}[Z_{(0,a)}(E')]}{\mathrm{Im}[Z_{(0,a)}(E')]} = \frac{h'}{c'} - \frac{r'}{c'}\frac{a^2}{2} < \frac{c}{2r} - \left(\frac{r}{c} + \frac{1}{c^2}\right)\frac{a^2}{2} =: Q(E,a). \tag{3.4}$$

Choose a_0 large enough such that

$$Q(E,a_0) < -\frac{\mathrm{Re}[Z_{(0,a)}(E)]}{\mathrm{Im}[Z_{(0,a)}(E)]} = \frac{h}{c} - \frac{r}{c}\frac{a^2}{2}. \tag{3.5}$$

So if $a > a_0$, inequalities (3.4) and (3.5) imply that for any for any $\nu_{(0,a)}$-semistable subobject $E' \subset E$, we have $\phi_{(0,a_0)}(E') < \phi_{(0,a)}(E)$. Therefore, E is $\nu_{(0,a)}$-stable. $\square$

The next lemma describes the position of the wall that bounds the large volume limit for a torsion free sheaf.

Lemma 3.3.2. *Given a μ_H-stable torsion free sheaf with $v_H(E) = (r,c,h)$ such that $r > H^n$. Then E is $\nu_{(b,a)}$-stable for any $a > 0$ and $c/r - \delta < b < c/r$ where*

$$\delta = \frac{H^n}{r(r - H^n)}.$$

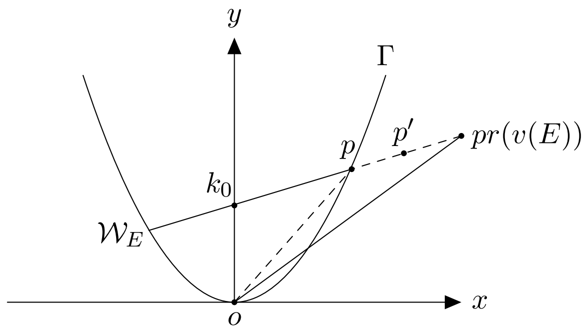

Figure 3.2: The first wall for E

Proof. Similar to the proof of Lemma 3.3.1, we may assume that $c > 0$. Lemma 3.3.1 implies that E is $\nu_{(b,a)}$-(semi)stable where $b < c/r$ and $a \gg 0$, i.e. the point $k(b,a)$ is close to the origin. Let $\mathcal{W}_E$ be the wall that bounds the chamber containing these stability conditions, see Proposition 2.3.16. Suppose the wall $\mathcal{W}_E$ intersects the y-axis at the point $k_0 := k(0,a_0)$. Assume E' is a destabilising subobject on the wall, so we have the short exact sequence

$$E' \hookrightarrow E \twoheadrightarrow E''. \tag{3.6}$$

We can assume E' is $\nu_{(0,a_0)}$-stable. Let $v_H(E') = (r', c', h')$. Since E' and E'' have the same phase with respect to weak stability condition $\nu_{(0,a_0)}$, we have

$$0 < \mathrm{Im}[Z_{(0,a_0)}(E')] \ \text{ and } \ 0 < \mathrm{Im}[Z_{(0,a_0)}(E'')] \ \Rightarrow \ 0 < c' < c. \tag{3.7}$$

Taking cohomology from the short exact sequence (3.6) implies that $H^{-1}(E') = 0$, so E' is a sheaf. Since E is $\nu_{(0,a)}$-stable where $a \gg 0$, we have

$$\phi_{(0,\infty)}(E') < \phi_{(0,\infty)}(E) \quad \Rightarrow \quad \frac{-r'}{c'} < \frac{-r}{c} \tag{3.8}$$

which, in particular, implies $r' \neq 0$. Let p be the point with positive x-coordinate where the wall $\mathcal{W}_E$ and the curve Γ intersect, see Figure 3.2. Assume the slope of the line segment $\overline{op}$ is $1/b_1$.

If $h' \neq 0$, Lemma 3.2.2 implies that the point $p' \coloneqq pr(v_H(E'))$ is on the line along the wall $\mathcal{W}_E$. Since E' is $\nu_{(0,a_0)}$-stable, the point p' cannot be above the curve Γ. Thus the slope of the line segment $\overline{op'}$ is less than the slope of $\overline{op}$, see Figure 3.2. Therefore,

$$\frac{r'}{c'} \leq \frac{1}{b_1} \tag{3.9}$$

If $h' = 0$, then the slope of the wall $\mathcal{W}_E$ is equal to r'/c' which is obviously less than the slope of $\overline{op}$, ao again inequality (3.9) is satisfied. If $0 < r' < r$, then inequality (3.8) shows that

$$\frac{c}{r} - \frac{c'}{r'} = \frac{cr' - c'r}{rr'} = \frac{cH^n\mathrm{ch}_0(E') - c'H^n\mathrm{ch}_0(E)}{rr'} \geq \frac{H^n}{r(r - H^n)}. \tag{3.10}$$

If $r \leq r'$, then inequality (3.7) implies

$$\frac{c'}{r'} \leq \frac{c}{r} \quad \Rightarrow \quad \frac{c}{r} - \frac{c'}{r'} \geq \frac{1}{r} \geq \frac{H^n}{r(r - H^n)}. \tag{3.11}$$

Therefore, in any case inequality (3.9) gives

$$\frac{c}{r} - b_1 \geq \frac{c}{r} - \frac{c'}{r'} \geq \frac{H^n}{r(r - H^n)} =: \delta. \tag{3.12}$$

This means $b_1 \leq c/r - \delta$ and the claim follows. □

Remark 3.3.3. In the proof of Lemma 3.3.2, inequality (3.10), we assumed that E has a subobject with the maximum possible slope, but if it is not the case, then we can find a better bound for the first wall, see [Sun16] for further investigation. In this paper, the author uses semicircle walls and his method is different from the one that we present here.

We also need to find some weak stability conditions where the shift of a μ_H-stable locally free sheaf is stable.

Lemma 3.3.4. *Let E be a μ_H-stable vector bundle with $v_H(E) = (r, c, h)$. Then $E[1]$ is $\nu_{(b,a)}$-stable for $b = c/r$ and $a > 0$.*

Proof. By definition, $E[1] \in \mathcal{A}(b)$ and $\mathrm{Im}[Z_{(b,a)}(E)] = 0$. Therefore, E is $\nu_{(b,a)}$-semistable of phase one. Assume for a contradiction that $E[1]$ is not $\nu_{(b,a)}$-stable.

Let F_1 be the $\nu_{(b,a)}$-stable subobject of $E[1]$ and F_2 be the quotient

$$F_1 \hookrightarrow E[1] \twoheadrightarrow F_2. \tag{3.13}$$

Taking cohomology implies that $H^0(F_2) = 0$, so $F_2 = E'[1]$ for a torsion free sheaf E'. There is a short exact sequence in $\mathcal{A}(b)$

$$H^{-1}(F_1)[1] \hookrightarrow F_1 \twoheadrightarrow H^0(F_1).$$

Thus, $H^{-1}(F_1)[1]$ and $H^0(F_1)$ are $\nu_{(b,a)}$-semistable of phase one. Since we assumed F_1 is stable, one of them must be zero. Assume $H^{-1}(F_1) = 0$. By definition of the heart, rank $rk(F_1) = 0$, otherwise, $\mu_H(H^0(F_1)) < b$ and $H^0(F_1)$ cannot be of phase one. Therefore, $\mathrm{Im}[Z_{(b,a)}(F_1)] = \mathrm{ch}_1(F_1).H^{n-1} = 0$. Taking cohomology from the short exact sequence (3.13) gives the following exact sequence of coherent sheaves

$$0 \to E \to E' \to H^0(F_1) \to 0.$$

Since $H^0(F_1)$ is of codimension at least 2 and E is a locally free sheaf, we have $E' \subset E^{\vee\vee} = E$ which implies $H^0(F_1) = 0$, a contradiction.

If $H^{-1}(F_1) \neq 0$, then we have the following exact sequence of coherent sheaves

$$0 \to H^{-1}(F_1) \to E \to E' \to 0.$$

Since $\mathrm{Im}\big[Z_{(b,a)}\big(H^{-1}(F_1)\big)\big] = \mathrm{Im}[Z_{(b,a)}(E')] = 0$, the sheaves E' and $H^{-1}(F_1)$ have the same slope as E, which contradicts the μ_H-stability of E. $\square$

One can apply the same argument as in Lemma 3.3.2 for the shift of a locally free sheaf to get the following result.

Lemma 3.3.5. *Let E be a μ_H-stable vector bundle with $v_H(E) = (r, c, h)$. Then $E[1]$ is $\sigma_{(b,a)}$-stable for any $a > 0$ and $c/r \leq b < c/r + \delta$ where*

$$\delta = \frac{H^n}{r(r - H^n)}.$$

Finally, we can prove the main result.

Proof of Theorem 3.1.1. The first step is to find a weak stability condition such that the objects E and $E(-lH)[1]$ have the same phase. So we first need that the line through the points $pr(v_H(E))$ and $pr(v_H(E(-lH)))$ intersect the parabola Γ at two points $q_1 = (2/b_1, 2/b_1^2)$ and $q_2 = (2/b_2, 2/b_2^2)$, see Figure 3.3.

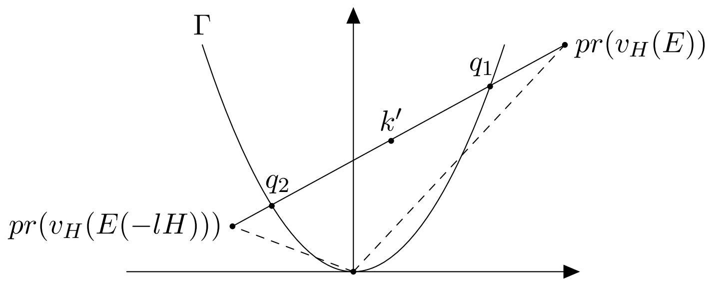

Figure 3.3: Stability of the restricted bundle

Write $v_H(E) = (r, c, h)$. One can easily compute that

$$b_1, b_2 = \frac{c}{r} - \frac{l}{2} \pm \sqrt{\frac{l^2}{4} - \frac{\overline{\Delta}_H(E)}{r^2}}.$$

Assume the point $k' := k(b', a')$ is on the line segment $\overline{q_1 q_2}$. Lemma 3.3.2 and 3.3.5 imply that E and $E(-lH)[1]$ are $\nu_{(b',a')}$-stable if

$$\left| \frac{c - rl}{r} - b_2 \right| = \left| \frac{c}{r} - b_1 \right| = \frac{l}{2} - \sqrt{\frac{l^2}{4} - \frac{\overline{\Delta}_H(E)}{r^2}} < \delta.$$

In other words,

$$\frac{\overline{\Delta}_H(E)}{r^2 \delta} + \delta < l. \tag{3.14}$$

In this case, E and $E(-lH)[1]$ are $\nu_{(b',a')}$-stable of the same phase, so their extension $i_*E|_C$ for a divisor $C \in |lH|$,

$$E \hookrightarrow i_*E|_C \twoheadrightarrow E(-lH)[1],$$

is $\nu_{(b',a')}$-semistable. For a positive number $\epsilon > 0$,

$$\phi_{(b',a')}(E) < \phi_{(b',a')}(i_*E|_C).$$

So the uniqueness of Jordan-Hölder factors implies that push-forward of the restriction $i_*E|_C$ is $\nu_{(b',a'+\epsilon)}$-stable if ϵ is sufficiently small. Let F be a non-trivial subsheaf of $E|_C$ on C. By definition, i_*F is a subobject of $i_*E|_C$ in $\mathcal{A}(b')$, so $\nu_{(b',a'+\epsilon)}$-stability of $i_*E|_C$ implies that

$$\phi_{(b',a'+\epsilon)}(i_*F) < \phi_{(b',a'+\epsilon)}(i_*E|_C) \quad \Rightarrow \quad \frac{\mathrm{ch}_2(i_*F).H^{n-2}}{\mathrm{ch}_1(i_*F).H^{n-1}} < \frac{\mathrm{ch}_2(i_*E|_C).H^{n-2}}{\mathrm{ch}_1(i_*E|_C).H^{n-1}}.$$

Thus, the claim follows by Hirzebruch-Riemann-Roch Formula and Definition 2.2.1. □

Remark 3.3.6. In Theorem 3.1.1, we only considered the restriction of μ_H-stable vector bundles to divisors in the linear systems of multiples of H, but similar methods can be used to investigate the stability of the restriction of a vector bundle to other divisors on the variety, see [Kop18] for the surface case.

If the variety X or the vector bundle E satisfy some extra conditions, then we can find a better bound for the wall that bounds the large volume limit which finally leads to a stronger restriction theorem.

Proof of Theorem 3.1.2. Assume the polarised variety (X, H) satisfies $H^n | H'H^{n-1}$ for any divisor H' on X. In the proof of Lemma 3.3.2, for the case $r' < r$ we have

$$\frac{c}{r} - \frac{c'}{r'} = \frac{cr' - c'r}{rr'} \geq \frac{(H^n)^2}{r(r - H^n)},$$

and for the case $r \leq r'$,

$$\frac{c'}{r'} \leq \frac{c - H^n}{r} \quad \Rightarrow \quad \frac{c}{r} - \frac{c'}{r'} \geq \frac{H^n}{r} \geq \frac{(H^n)^2}{r(r - H^n)}.$$

Therefore, Lemma 3.3.2 and 3.3.5 are valid for

$$\delta = \frac{(H^n)^2}{r(r - H^n)}.$$

Thus, proof of Theorem 3.1.1, inequality (3.14) implies that the restriction $E|_C$ is $\mu_{H|_C}$stable if

$$\frac{\overline{\Delta}_H(E)(r - H^n)}{r(H^n)^2} + \frac{(H^n)^2}{r(r - H^n)} < l.$$

$\square$

Chapter 4

Mukai's Program

4.1 Introduction

In this chapter, we consider the problem of reconstructing a K3 surface from a curve on that surface. The main result is the following, which extends and completes a program proposed by Mukai in [Muk01, Section 10].

Theorem 4.1.1. *Let (X, H) be a polarised K3 surface with Pic$(X) = \mathbb{Z}.H$. Let C be any curve in the linear system $|H|$ of genus $g = 11$ or $g \geq 13$. Then X is the unique K3 surface of Picard rank one and genus g containing C, and can be reconstructed as a Fourier-Mukai partner of a certain Brill-Noether locus of vector bundles on C.*

Note that any K3 surface of Picard rank one has a canonical polarisation and therefore a well-defined genus. To be more precise, we need to consider two different cases. Let $M_C(r, d, h)$ be the Brill-Noether locus of slope semistable rank r-vector bundles on the curve C having degree d and possessing at least h linearly independent global sections, and $M_C^{\mathrm{st}}(r, d, h)$ be the Brill-Noether locus of slope stable vector bundles. Let $M_{X,H}(\bar{v})$ be the moduli space of H-Gieseker semistable sheaves with Mukai vector $\bar{v}$ on X.

- Case (A): If the genus $g = rs + 1$ for two integers $r \geq 2$ and $s \geq \max\{r, 5\}$, we consider the Brill-Noether locus BN$:= M_C(r, 2rs, r + s)$ and the moduli space $M_{X,H}(\bar{v})$ where $\bar{v} = (r, H, s)$.
- Case (B): If the genus $g = p + 1$ for some odd number $p \geq 13$, we consider the Brill-Noether locus BN$:= M_C^{\mathrm{st}}(4, 4p, p + 4)$ and the moduli space $M_{X,H}(\bar{v})$ where $\bar{v} = (4, 2H, p)$.

In both cases, $\bar{v}$ is primitive with $\bar{v}^2 = 0$, hence $M_{X,H}(\bar{v})$ is K3 surface as well.

Theorem 4.1.2. *Let (X, H) be a polarised K3 surface with Pic$(X) = \mathbb{Z}.H$ and C is any curve in the linear system $|H|$. Then, in both cases (A) and (B), we have an isomorphism*

$$\psi \colon M_{X,H}(\bar{v}) \to BN \tag{4.1}$$

with BN as defined above, which sends a bundle E on X to its restriction $E|_C$,

In other words, special vector bundles on the curve C, which have an unexpected number of global sections, are the restriction of vector bundles on the surface X. This is analogous to the case of line bundles, where a well-known theorem by Green and Lazarsfeld [GL87] says that the Clifford index of a non-Clifford general curve on a K3 surface can be computed by the restriction of a line bundle on the surface.

In both cases (A) and (B), there exists a Brauer class $\alpha \in Br(\mathrm{BN})$ and a universal $(1 \times \alpha)$-twisted sheaf $\mathcal{E}$ on $C \times (\mathrm{BN}, \alpha)$. Define $v' \in H^*\big(\mathrm{BN}, \mathbb{Z}\big)$ to be Mukai vector of $\mathcal{E}|_{p \times (\mathrm{BN},\alpha)}$ for a point p on the curve C (see [HS05] for definition in case $\alpha \neq 1$).

Theorem 4.1.3. *Let (X, H) be a polarised K3 surface with $Pic(X) = \mathbb{Z}.H$ of genus $g = 11$ or $g \geq 13$, and let C be any curve in the linear system $|H|$. Then any K3 surface of Picard rank one and genus g which contains the curve C is isomorphic to the moduli space $M_{(BN,\alpha),H'}(v')$ for a generic polarisation H' on BN.*

Combining Theorems 4.1.2 and 4.1.3 gives Theorem 4.1.1.

4.1.1 Previous work

Let $\mathcal{F}_g$ be the moduli space of polarised K3 surfaces (X, H) where H is a primitive ample line bundle on X and $H^2 = 2g - 2$. This space is a quasi-projective variety of dimension 19. Let $\mathcal{P}_g$ be the moduli space of pairs (X, C) such that $(X, H) \in \mathcal{F}_g$ and C is a smooth curve in the linear system $|H|$. Therefore, its dimension is $19 + g$. Finally, let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g. Its dimension is $3g - 3$. The space $\mathcal{P}_g$ has natural projections to $\mathcal{F}_g$ and $\mathcal{M}_g$ which we denote by ϕ_g and m_g, respectively;

$$\begin{tikzcd} & \mathcal{P}_g \arrow[dl, "m_g"'] \arrow[dr, "\phi_g"] & \\ \mathcal{M}_g & & \mathcal{F}_g \end{tikzcd}$$

The map m_g is dominant for $g \leq 11$ and $g \neq 10$ [Muk88]. In [CLM93, Theorem 5], Ciliberto, Lopez and Miranda proved that for $g \geq 11$ and $g \neq 12$, the map m_g is birational onto its image. For the exceptional cases $g = 10$ or $g = 12$, the map m_g is neither dominant nor generically finite [Muk01].

In [Muk01], Mukai introduced a geometric program to find the rational inverse of m_g where $g = 2s + 1$ and $s \geq 5$ odd. His idea to reconstruct the K3 surface is as follows. Let C be a general curve on the image of m_g. Consider the Brill-Noether locus $M_C(2, K_C, s + 2)$ of stable rank 2-vector bundles on the curve C with canonical determinant and possessing at least $s + 2$ linearly independent global sections. Then $M_C(2, K_C, s + 2)$ is a K3 surface and the K3 surface containing the curve C can be obtained uniquely as a Fourier-Mukai transform of the Brill-Noether locus.

This program was completely proved by him in [Muk96] for $g = 11$. The key idea is that all vector bundles in the Brill-Noether locus $M_C(2, K_C, 7)$ are the restriction of vector bundles on the surface. He first considers a point $(X', C') \in \mathcal{P}_g$ of a special type and shows that the Brill-Noether locus $M_{C'}(2, K_{C'}, 7)$ is isomorphic to X'. Indeed, he proves that both surfaces are isomorphic to the moduli space $M_{X',H'}(\bar{v})$ where $\bar{v} = (2, H, 5)$. Given a general pair $(X, C) \in \mathcal{P}_g$, the Brill-Noether locus $M_C(2, K_C, 7)$ is a flat deformation of $M_{C'}(2, K_{C'}, 7)$ and has expected dimension. Thus, it is again a K3 surface and the original K3 surface can be obtained as an appropriate Fourier-Mukai transform of it.

Arbarello, Bruno and Sernesi [ABS14] generalised this strategy to higher genera. They proved that for a general pair $(X, C) \in \mathcal{P}_g$ where $g = 2s + 1 \geq 11$, there is a unique irreducible component V_C of $M_C(2, K_C, s + 2)$ such that $V_{C_{\mathrm{red}}}$ is a K3 surface isomorphic to the moduli space $M_{X,H}(\bar{v})$ where $\bar{v} = (2, H, s)$. Then they showed that the original K3 surface can be reconstructed using this component whenever $g \equiv 3 \mod 4$.

In this chapter, without any deformation argument, we show that for a general pair $(X, C) \in \mathcal{P}_g$, when $g = 2s + 1 \geq 11$, the Brill-Noether locus $M_C(2, K_C, s + 2)$ is isomorphic to the moduli space $M_{X,H}(2, H, s)$, and when $g = p + 1$ for some odd number $p \geq 13$, the Brill-Noether locus $M_C(4, 2K_C, 4 + p)$ is isomorphic to the moduli space $M_{X,H}(4, 2H, p)$ which is again a K3 surface. As a result, we prove the uniqueness of the K3 surface of Picard rank one which contains the curve C of genus $g = 11$ or $g \geq 13$.

4.1.2 Strategy of the proof

We prove Theorem 4.1.2 by wall-crossing for the push-forward of semistable vector bundles on the curve C, with respect to Bridgeland stability conditions on the bounded derived category $\mathcal{D}(X)$ of X. There exists a region in the space of stability conditions where the Brill-Noether behaviour of *stable* objects is completely controlled by the nearby *Brill-Noether wall*. This wall destabilises objects with non-zero global sections, and arguments similar to [Bay16b] show that the Brill-Noether loci are mostly of expected dimension. Our first key result, Proposition 4.3.4, gives an extension to *unstable* objects: it gives a bound on the number of global sections in terms of their *mass*, i.e. the length of their Harder-Narasimhan polygon.

Consequently, we only need a polygon that circumscribes this Harder-Narasimhan polygon on the left, to bound the number of global sections. For any coherent sheaf, there exists a chamber which is called the Gieseker chamber, where the notion of Bridgeland stability coincides with the old notion of Gieseker stability. Unlike the case of push-forward of line bundles considered in [Bay16b], the Brill-Noether wall is not adjacent to the Gieseker chamber for the push-forward of semistable vector bundles F of higher ranks on the curve C. However, the wall that bounds the Gieseker chamber provides an extremal polygon which contains the Harder-Narasimhan polygon, see e.g. Lemma 4.4.3. Combined with Proposition 4.3.4, this gives a bound on the number of global sections of vector bundles on the curve C; the proof also shows that the bound is sharp if and only if the vector bundle F is the restriction of a vector bundle on the surface.

4.2 Bridgeland stability conditions on K3 surfaces

In this section, we give a brief review of a two-dimensional family of Bridgeland stability conditions on the bounded derived category of coherent sheaves on a K3 surface. The main references are [Bri07, Bri08].

Suppose X is an algebraic K3 surface over $\mathbb{C}$, i.e. a complete non-singular variety of dimension two such that

$$\omega_X \simeq \mathcal{O}_X \quad \text{and} \quad H^1(X, \mathcal{O}_X) = 0.$$

For instance, a smooth quartic $X \subset \mathbb{P}^3$ is a K3 surface. Let H be an ample line bundle on X. We always assume $\mathrm{Pic}(X) = \mathbb{Z}.H$. We denote by $\mathcal{D}(X) = \mathcal{D}^b\mathrm{Coh}(X)$ the bounded derived category of coherent sheaves on X. The Mukai vector of an object $E \in \mathcal{D}(X)$ is an element of the lattice $\mathcal{N}(X) = \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z} \cong \mathbb{Z}^3$, defined via

$$v(E) = \big(r(E), c(E)H, s(E)\big) = \mathrm{ch}(E)\sqrt{\mathrm{td}(X)} \in H^*(X, \mathbb{Z}),$$

where $\mathrm{ch}(E)$ is the Chern character of E. The Mukai bilinear form

$$\left\langle v(E), v(E')\right\rangle = c(E)c(E')H^2 - r(E)s(E') - r(E')s(E)$$

makes $\mathcal{N}(X)$ into a lattice of signature $(2,1)$. The Riemann-Roch theorem implies that this form is the negative of the Euler form, defined as

$$\chi(E,E') = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}^i_X(E,E') = -\left\langle v(E), v(E')\right\rangle.$$

Note that the Euler form $\chi(-,-)$ defines a bilinear form on the Grothendieck group $K(X)$ which descends to a non degenerate form on the lattice

$$\mathcal{N}(X) = K(X)/K(X)^{\perp},$$

where $K(X)^{\perp}$ is the left-radical. For any pair of objects E and E' of $\mathcal{D}(X)$, Serre duality gives isomorphisms

$$\mathrm{Hom}^i_X(E,E') \cong \mathrm{Hom}^{2-i}_X(E',E)^*.$$

If the objects E and E' lie in the heart of a bounded t-structure on $\mathcal{D}(X)$, such as the category of coherent sheaves, then

$$\mathrm{Hom}^i_X(E,E') = 0 \quad \text{if} \;\; i<0 \;\; \text{or} \;\; i>2.$$

Suppose the object E is stable with respect to a (weak) stability condition, or it is μ_H-stable. Since E does not have any non-trivial subobject with the same phase (slope),

$$\mathrm{Hom}_X(E,E) = \mathrm{Hom}^2_X(E,E)^* = \mathbb{C}.$$

This implies

$$v(E)^2 + 2 = \mathrm{Hom}^1_X(E,E) \geq 0, \tag{4.2}$$

which is stronger than the Bogomolov inequality. An object $E \in \mathcal{D}(X)$ is called spherical if $\mathrm{Hom}^i_X(E,E) = \mathbb{C}$ for $i = 0,2$ and it is zero otherwise. Therefore, the Mukai vector of a spherical object lies in the root system

$$\Delta(X) \coloneqq \{\delta \in \mathcal{N}(X) \colon \; \delta^2 = -2\}.$$

Conversely, for every $\delta = (r, cH, s) \in \Delta(X)$ with $r > 0$, there exists a μ_H-stable spherical vector bundle with Mukai vector δ; see [Kul90] and [Muk87, Proposition 3.14].

All stability functions that we consider in this chapter, factor through the surjection $K(X) \twoheadrightarrow \mathcal{N}(X)$. Given a pair $(b,w) \in \mathbb{R} \times \mathbb{R}^{>0}$ in the upper half plane, the stability function $Z_{(b,w)} \colon \mathcal{N}(X) \to \mathbb{C}$ is defined as

$$Z_{(b,w)}(E) = bc(E)H^2 - s(E) - \frac{H^2}{2} r(E)(b^2 - w^2) + i\big(c(E) - br(E)\big).$$

If $H^2w^2 > 2$, then by replacing H^2w^2 with a^2-2, we get the function $Z_{(b,a)}$ which has been studied in Chapter 2. Recall that the heart $\mathcal{A}(b) = \langle \mathcal{T}^b, \mathcal{F}^b[1]\rangle$ where

$$\mathcal{T}^b = \langle E \colon E \text{ is } \mu_H\text{-semistable sheaf with } \mu_H(E) > b\rangle,$$

$$\mathcal{F}^b = \langle E \colon E \text{ is } \mu_H\text{-semistable sheaf with } \mu_H(E) \leq b \rangle.$$

Theorem 4.2.1 ([Bri08]). *Let (X, H) be a polarised K3 surface with Pic$(X) = \mathbb{Z}.H$. Then the pair $\sigma_{(b,w)} = \big(\mathcal{A}(b), Z_{(b,w)}\big)$ defines a Bridgeland stability condition on $\mathcal{D}(X)$ if Re$[Z_{(b,w)}(\delta)] > 0$ for all roots $\delta \in \Delta(X)$ of the form (r, brH, s) with $r > 0$. Also, the family of stability conditions $\sigma_{(b,w)}$ varies continuously as the pair (b, w) varies in $\mathbb{H}$.*

Proof. It is enough to show that for any non-zero object $E \in \mathcal{A}(b)$, the complex number $Z_{(b,w)}(E)$ lies in the upper half plane or negative real line. The existence of Harder-Narasimhan filtration can be verified with the same arguments as weak stability conditions in Chapter 2.

If E is a torsion free sheaf, then by definition $\mu_H(E) > b$ and $\mathrm{Im}[Z_{(b,w)}(E)] > 0$. If E is a torsion sheaf and $c(E) \neq 0$, then E is supported on a curve, so $c(E) > 0$ and again imaginary part is positive. If $c(E) = 0$ for a torsion sheaf E, then it is a skyscraper sheaf and $s(E) > 0$ which means $\mathrm{Re}[Z_{(b,w)}(E)] < 0$. Similarly, if E is a μ_H-stable torsion free sheaf with $\mu_H(E) < b$, the complex number $Z_{(b,w)}(E[1])$ lies in the upper half plane. If $\mu_H(E) = b$, then

$$\mathrm{Re}[Z_{(b,w)}(E[1])] = -\frac{v(E)^2}{2r(E)} - \frac{H^2 w^2 r(E)}{2},$$

which is negative if $v(E)^2 \neq -2$, so the claim follows. □

Note that the Bridgeland stability condition $\sigma_{(b,w)}$ up to the action of $\tilde{\mathrm{GL}}^+(2, \mathbb{R})$, is the same as the stability condition defined in [Bri08, Section 6].

For an object $E \in \mathcal{A}(b)$, we have $Z_{(b,w)}(E) = m(E) \exp(i\pi\phi_{(b,w)}(E))$ where $m(E) > 0$ and

$$\phi_{(b,w)}(E) = \frac{1}{\pi} \tan^{-1}\left(-\frac{\mathrm{Re}[Z_{(b,w)}(E)]}{\mathrm{Im}[Z_{(b,w)}(E)]}\right) + \frac{1}{2} \in (0, 1].$$

Consider the projection

$$pr \colon \mathcal{N}(X) \setminus \{s = 0\} \to \mathbb{R}^2 \ \ , \ \ pr(r, cH, s) = \left(\frac{c}{s}, \frac{r}{s}\right).$$

Given a pair $(b, w) \in \mathbb{R} \times \mathbb{R}^{>0}$, the kernel of the stability function $Z_{(b,w)}$ lies on the line inside the negative cone in $\mathcal{N}(X) \otimes \mathbb{R} \cong \mathbb{R}^3$ spanned by the vector $\big(2, 2bH, H^2(b^2 + w^2)\big)$. Its projection is denoted by

$$k(b, w) := pr\big(\mathrm{Ker} Z_{(b,w)}\big) = \left(\frac{2b}{H^2(b^2 + w^2)}, \frac{2}{H^2(b^2 + w^2)}\right).$$

Thus to any stability condition $\sigma_{(b,w)}$ we associate a point $k(b, w) \in \mathbb{R}^2$. The two dimensional family of stability conditions of the form $\sigma_{(b,w)}$ is continuously parametrised by the space

$$V(X) := \big\{k(b, w) \colon \text{ the pair } \big(\mathcal{A}(b), Z_{(b,w)}\big) \text{ is a stability condition on } \mathcal{D}(X)\big\} \subset \mathbb{R}^2$$

with the standard topology on $\mathbb{R}^2$.

Lemma 4.2.2. *We have*

$$V(X) = \left\{(x, y) \in \mathbb{R}^2 \colon \ y > \frac{H^2}{2}\, x^2 \right\} \setminus \bigcup_{\delta \in \Delta(X)} I_\delta$$

where I_δ is the closed line segment that connects $pr(\delta)$ to p_δ which is the intersection point of the parabola $y = \dfrac{H^2}{2}x^2$ with the line through the origin and $pr(\delta)$, see Figure 4.1.

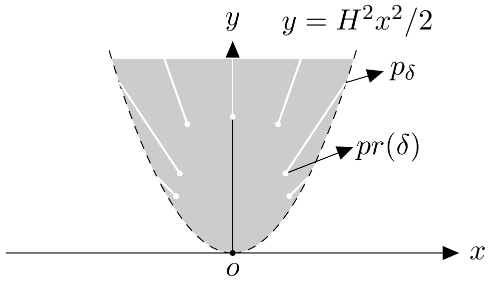

Figure 4.1: The grey area is a 2-dimensional subspace of stability conditions.

Proof. By definition, the point $k(b, w)$ is above the parabola and for every point (x, y) above the parabola, there exists a unique pair $(b, w) \in \mathbb{H}$ such that $k(b, w) = (x, y)$. If $k(b, w)$ is on a line passing through the origin and $pr\big(\delta = (r, cH, s)\big)$, then $b = c/r$ and $\mathrm{Im}[Z_{(b,w)}(\delta)] = 0$. One can show if $k(b, w)$ is on the line segment I_δ and $r > 0$, then $\mathrm{Re}[Z_{(b,w)}(\delta)] \leq 0$, and the claim follows from Theorem 4.2.1. $\square$

Note that the point $k(b, w)$ is on a line with equation $x = by$. As w gets larger, the point $k(b, w)$ gets closer to the origin. In the figures, by abuse of notation, we denote the point $k(b, w) \in V(X)$ by the corresponding stability condition $\sigma_{(b,w)}$.

The following lemma ensures non-existence of projection of roots in some critical areas. We denote by γ_n the point $\big(1/n, H^2/(2n^2)\big)$ on the parabola for any $n \in \mathbb{Q}$.

Lemma 4.2.3. *For any positive number $n \in \dfrac{1}{2}\mathbb{N}$, define*

$$U_n := \left\{ (x, y) \in \mathbb{R}^2 \colon 0 < |x| < \frac{1}{n} \text{ and } \frac{H^2}{2n}\,|x| < y \right\}.$$

If $n \leq \dfrac{H^2}{2}$, then there is no projection of roots $pr(\delta)$ in U_n.

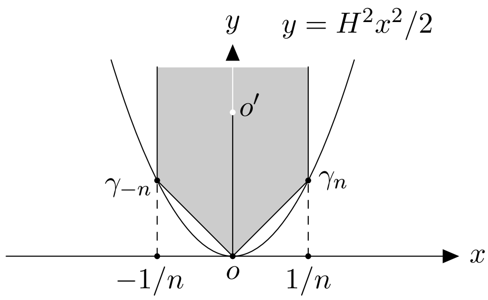

Figure 4.2: No projection of roots in the grey area U_n

Proof. Assume for a contradiction that $pr(\delta = (\tilde{r}, \tilde{c}H, \tilde{s})) \in U_n$, then

$$0 < \frac{H^2}{2n}\left|\frac{\tilde{c}}{\tilde{s}}\right| < \left|\frac{\tilde{r}}{\tilde{s}}\right|, \tag{4.3}$$

which implies $\left|\tilde{c}^2 H^2\right| < |2n\tilde{r}\tilde{c}|$. By assumption $\delta^2 = \tilde{c}^2 H^2 - 2\tilde{r}\tilde{s} = -2$, so

$$0 < \left|\tilde{s} - \frac{1}{\tilde{r}}\right| < |n\tilde{c}|\,. \tag{4.4}$$

Moreover,

$$0 < \left|\frac{\tilde{c}}{\tilde{s}}\right| < \frac{1}{n} \quad \Rightarrow \quad 0 < |n\tilde{c}| < |\tilde{s}|\,. \tag{4.5}$$

If $n \in \mathbb{N}$, there is no triple $(\tilde{r}, \tilde{c}, \tilde{s}) \in \mathbb{Z}^3$ that satisfies both inequalities (4.4) and (4.5) and if $n \in \frac{1}{2}\mathbb{N}$, the only possible case is $\tilde{r} = \pm 1$. But we assumed $2n \leq H^2$ and inequality (4.3) implies $0 < |\tilde{c}| < 1$, a contradiction. □

Remark 4.2.4. Note that if the point $pr\big(\delta = (\tilde{r}, \tilde{c}H, \tilde{s})\big) = (\tilde{c}/\tilde{s}, \tilde{r}/\tilde{s})$ is on the y-axis, then $\tilde{c} = 0$. Since $\delta^2 = -2\tilde{r}\tilde{s} = -2$, we have $\tilde{r} = \tilde{s} = \pm 1$ and $pr(\delta) = (0,1) = pr(v(\mathcal{O}_X))$. This point is denoted by o' in Figure 4.2.

Given three positive numbers $m, n, \epsilon \in \frac{1}{2}\mathbb{N}$ such that $m < n$, the point on the line segment $\overline{\gamma_m \gamma_n}$ with the x-coordinate $1/(m+\epsilon)$ is denoted by $q'_{m,n,\epsilon}$. Also, the point where the line segments $\overline{\gamma_m \gamma_n}$ and $\overline{o\gamma_{n-\epsilon}}$ intersect is denoted by $q_{m,n,\epsilon}$, see Figure 4.3. One can define similar points for the triple $(-m, -n, -\epsilon)$.

For two points $q_1, q_2 \in \mathbb{R}^2$, we denote by $[\overline{q_1 q_2}]$ the closed line segment which contains both q_1 and q_2. The open line segment which contains neither q_1 nor q_2 is denoted by $(\overline{q_1 q_2})$.

Lemma 4.2.5. *Let m, n, $\epsilon \in \frac{1}{2}\mathbb{N}$ be three positive numbers such that*

$$\epsilon + \frac{1}{2} < n \leq \frac{H^2}{2} \quad \text{and} \quad m < \frac{2\epsilon}{2\epsilon + 1} n - \epsilon.$$

Then there is no projection of roots in the grey area in Figure 4.3 and on the open line segments $(\overline{q_{m,n,\epsilon}\, q'_{m,n,\epsilon}})$ and $(\overline{q_{-m,-n,-\epsilon}\, q'_{-m,-n,-\epsilon}})$.

Proof. We show that the claimed region is contained in a suitable union of the U_k's. Given a number $k \in \frac{1}{2}\mathbb{N}$ where $m < k < n$, the point where the line segments $\overline{\gamma_m \gamma_n}$ and $\overline{o\gamma_k}$ intersect is denoted by γ'_k, see Figure 4.4.

The x-coordinate of the point γ'_k is

$$x_k = \frac{1/mn}{1/m + 1/n - 1/k}.$$

If $m + \epsilon \leq k \leq n - \epsilon - 1/2$, then

$$\frac{1}{mn}\left(k + \frac{1}{2}\right) + \frac{1}{k} < \frac{1}{m} + \frac{1}{n}$$

which implies $x_k < \dfrac{1}{k + 1/2}$, so the point $\gamma'_k \in U_{k+1/2}$. Therefore, the grey region in Figure 4.3 is contained in $\displaystyle\bigcup_{m+\epsilon \leq k \leq n} U_k$ where $k \in \frac{1}{2}\mathbb{N}$ and the claim follows from Lemma 4.2.3. □

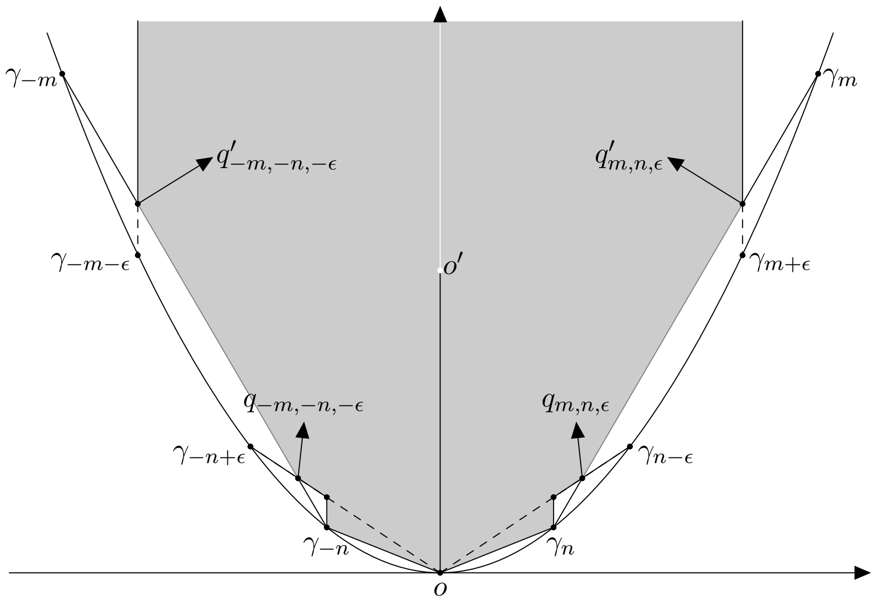

Figure 4.3: No projection of roots in the grey area

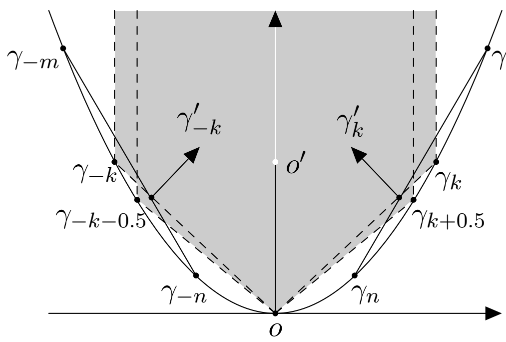

Figure 4.4: Two consecutive points

The 2-dimensional family of stability conditions parametrised by the space $V(X)$ admits a chamber decomposition for any object $E \in \mathcal{D}(X)$. Note that in the following, we do not assume $v(E)$ is primitive; in particular, E might be strictly semistable in the interior of a chamber.

Proposition 4.2.6. *Given an object $E \in \mathcal{D}(X)$, there exists a locally finite set of* walls *(line segments) in $V(X)$ with the following properties:*

(a) The $\sigma_{(b,w)}$-semistability or instability of E is independent of the choice of the stability condition $\sigma_{(b,w)}$ in any given chamber.
(b) When $\sigma_{(b_0,w_0)}$ is on a wall $\mathcal{W}_E$, i.e. the point $k(b_0,w_0) \in \mathcal{W}_E$, then E is strictly $\sigma_{(b_0,w_0)}$-semistable.
(c) If E is semistable in one of the adjacent chambers to a wall, then it is unstable in the other adjacent chamber.
(d) Any wall $\mathcal{W}_E$ is a connected component of $L \cap V(X)$, where L is a line that passes through the point $pr(v(E))$ if $s(E) \neq 0$, or that has a slope of $r(E)/c(E)$ if $s(E) = 0$.

Proof. The existence of a locally finite set of walls which satisfies properties (a), (b) and (c), is proved in [Bri08, Section 9], see also [Mac14] for the description of the walls.

One can apply the same argument as in the proof of Lemma 3.2.2 to deduce claim (d). $\square$

We use the next lemma to describe regions in $V(X)$ with no walls for a given object.

Lemma 4.2.7. *Given a stability condition $\sigma_{(b,w)}$ and an object $E \in \mathcal{D}(X)$ such that*

$$0 < \left|Im\,[Z_{(b,w)}(v(E))]\right| = \min\left\{ \left|Im\,[Z_{(b,w)}(v')]\right| : v' \in \mathcal{N}(X) \text{ and } Im[Z_{(b,w)}(v')] \neq 0 \right\},$$

then the stability condition $\sigma_{(b,w)}$ cannot be on a wall for the object E. In particular, if $v(E) = (r, cH, s)$ and $b_0 = m/n$ for some m, $n \in \mathbb{Z}$ such that $nc - mr = \pm 1$, then the stability condition $\sigma_{(b_0,w)}$ cannot be on a wall for E.

Proof. If the stability condition $\sigma_{(b,w)}$ is on a wall $\mathcal{W}_E$, then up to shift, we may assume $E \in \mathcal{A}(b)$. There are two objects E_1 and E_2 in $\mathcal{A}(b)$ which have the same phase as E and $E_1 \hookrightarrow E \twoheadrightarrow E_2$. Since $\mathrm{Im}\,[Z_{(b,w)}(E)] \neq 0$, we have $0 < \mathrm{Im}\,[Z_{(b,w)}(E_i)]$ for $i = 1, 2$ and

$$\mathrm{Im}\,[Z_{(b,w)}(E)] = \mathrm{Im}\,[Z_{(b,w)}(E_1)] + \mathrm{Im}\,[Z_{(b,w)}(E_2)].$$

This is a contradiction to our minimality assumption. If $b_0 = m/n$, then

$$\mathrm{Im}\,[Z_{(b_0,w)}(E)] = c - \frac{m}{n}r = \frac{\pm 1}{n}$$

which clearly satisfies the minimality condition. $\square$

Recall that the Hilbert polynomial of a sheaf E is defined as

$$P(E, m) := \frac{r(E)H^2}{2}m^2 + c(E)H^2 m + s(E) + r(E).$$

The reduced Hilbert polynomial is $p(E, m) := P(E, m)/\alpha(E)$ where $\alpha(E)$ is the leading coefficient of $P(E, m)$.

Definition 4.2.8. A coherent sheaf E on X is called H-Gieseker (semi)stable if E is pure, and for all proper non-trivial subsheaves $F \subset E$, one has $p(F, m) < (\leq)\ p(E, m)$ for $m \gg 0$.

The notion of slope stability for coherent sheaves on a curve (i.e. an integral separated scheme of dimension one and of finite type over $\mathbb{C}$) is also defined as follows.

Definition 4.2.9. A torsion free sheaf F on a curve C is slope (semi)stable if for all non-trivial subsheaves $F' \subset F$, we have

$$\frac{\chi(\mathcal{O}_C, F')}{\chi(\mathcal{O}_p, F')} < (\leq)\ \frac{\chi(\mathcal{O}_C, F)}{\chi(\mathcal{O}_p, F)},$$

where $\mathcal{O}_p$ is the skyscraper sheaf at the generic point p on the curve C and $\mathcal{O}_C$ is the structure sheaf of C.

Note that the number $\chi(\mathcal{O}_p, F)$ for the generic point p on the curve C is equal to the rank of the torsion free sheaf F. If we have a closed embedding $i\colon C \hookrightarrow X$ of the curve C into the surface X, then the adjoint functors $Li^* \dashv Ri_*$ give

$$\chi(\mathcal{O}_C, F) = \chi(\mathcal{O}_X, i_*F) = \mathrm{ch}_2(i_*F) = s(i_*F).$$

Therefore, the torsion free sheaf F on C is slope-(semi)stable if and only if i_*F is H-Gieseker (semi)stable.

A similar argument as [Bri08, Proposition 14.2] and [Mac14, Theorem 3.11] implies that there is a region in the subspace of stability conditions $V(X)$ where the notion of Bridgeland stability coincides with the old notion of Gieseker stability.

Lemma 4.2.10. *Let E be a coherent sheaf on X with Mukai vector $v(E) = (r, cH, s)$.*

(a) If E is H-Gieseker (semi)stable, then E is $\sigma_{(b,w)}$-(semi)stable for $w \gg 0$ and $b < c/r$ (or b arbitrary in case $r = 0$).
(b) If the sheaf E is of positive rank $r > 0$ and $\sigma_{(b,w)}$-(semi)stable for $b < c/r$ and $w \gg 0$, then E is H-Gieseker (semi)stable.
(c) If the sheaf E is of rank zero and $\sigma_{(b,w)}$-(semi)stable for some $(b, w) \in \mathbb{R} \times \mathbb{R}^{>0}$, then E is H-Gieseker (semi)stable.
(d) If E is a μ_H-stable locally free sheaf on the surface X, then $E[1]$ is $\sigma_{(b,w)}$-stable for $b = c/r$ and $w > 0$.

Proof. Suppose E is an H-Gieseker semistable sheaf. If E is a skyscraper sheaf $\mathcal{O}_x$, then claim (a) holds because $\mathcal{O}_x$ is of phase one and it is a simple object in $\mathcal{A}(b)$ by [Bri08, Lemma 6.3]. If $r \neq 0$ or $c \neq 0$, then similar to the argument of Lemma 3.3.1, we may assume $c > 0$, so it is enough to show that E is $\sigma_{(0,w)}$-(semi)stable for $w \gg 0$. Assume

$$E' \hookrightarrow E \twoheadrightarrow E''$$

is a destabilising sequence at the stability condition $\sigma_{(0,w)}$ for some $w > 0$. We may assume E' is $\sigma_{(0,w)}$-semistable. Taking cohomology gives a long exact sequence of sheaves

$$0 \to H^{-1}(E'') \to E' \xrightarrow{f} E \to H^0(E'') \to 0.$$

If $H^{-1}(E'') \neq 0$, then by definition of the heart $\mathcal{A}(0)$, $\mu_H(H^{-1}(E'')) \leq 0 < \mu_H(E')$. Therefore, $\mu_H(E') < \mu_H(\mathrm{Im} f) \leq \mu_H(E)$ which implies inequality (3.1) is satisfied, so the proof of Lemma 3.3.1 implies that for $w \gg 0$, we have $\varphi_{(0,w)}(E') < \varphi_{(0,w)}(E)$. If $H^{-1}(E') = 0$, then E' is a subsheaf of E, so claim (a) follows.

To prove claim (b), again we can assume $c > 0$ and the sheaf E is in the heart $\mathcal{A}(0)$. Assume for a contradiction that E is not H-Gieseker (semi)stable and

$$0 \to E' \to E \to E'' \to 0 \tag{4.6}$$

is a destabilising sequence of sheaves on X. We may assume E' is H-Gieseker semistable and $\mu_H(E') \geq \mu(E) > 0$. Moreover, we have $\mu_H^-(E'') \geq \mu_H^-(E) > 0$. Therefore, E' and E'' are both in the heart $\mathcal{A}(0)$. Assume $v(E') = (r', c'H, s')$. Since E is $\sigma_{(0,w)}$-(semi)stable for $w \gg 0$, we have

$$\phi_{(0,w)}(E') < (\leq) \phi_{(0,w)}(E) \quad \Rightarrow \quad \frac{s'}{c'} - \frac{r'}{c'}\frac{H^2w^2}{2} < (\leq)\ \frac{s}{c} - \frac{r}{c}\frac{H^2w^2}{2},$$

which gives $p(E', m) < (\leq)\ p(E, m)$ for $m \gg 0$, a contradiction.

If the sheaf E is of rank zero, then again we consider the destabilising sequence (4.6). In this case, the sheaves E' and E'' are torsion sheaves, so they are inside the heart $\mathcal{A}(b)$ for any $b \in \mathbb{R}$. Since the ordering of the phase $\phi_{(b,w)}$ is the same as the ordering of the reduced polynomial, the sheaf E is H-Gieseker (semi)stable. This proves claim (c). Claim (d) follows by the same argument as in Lemma 3.3.4. $\square$

4.3 An upper bound for the number of global sections

In this section, we study the *Brill-Noether wall* and introduce an upper bound for the number of global sections of objects in $\mathcal{D}(X)$ depending only on the geometry of their Harder-Narasimhan polygons at a certain limit point, see Proposition 4.3.4.

We always assume X is a smooth K3 surface with $\mathrm{Pic}(X) = \mathbb{Z}.H$. Given an object $E \in \mathcal{D}(X)$, we denote its Mukai vector by $v(E) = \big(r(E), c(E)H, s(E)\big)$.

Lemma 4.3.1. *Let $E \in \mathcal{A}(0)$ be a $\sigma_{(0,w)}$-semistable object with $\phi_{(0,w)}(E) < 1$. Then*

$$v(E)^2 \geq -2c(E)^2.$$

Proof. Let $0 = \tilde{E}_0 \subset \tilde{E}_1 \subset .... \subset \tilde{E}_{n-1} \subset \tilde{E}_n = E$ be the Jordan-Hölder filtration of E with respect to the stability condition $\sigma_{(0,w)}$. Since the stable factors $E_i = \tilde{E}_i/\tilde{E}_{i-1}$ have the same phase as E, we have

$$\mathrm{Im}[Z_{(0,w)}(E_i)] = c(E_i) > 0.$$

Therefore, the length of the filtration n is at most $c(E)$. Given two factors E_i and E_j, we know $\mathrm{Hom}_X(E_i, E_j) = 0$ if $E_i \ncong E_j$ and $\mathrm{Hom}_X(E_i, E_i) = \mathbb{C}$. Thus, for any $0 < i, j \leq n$,

$$\langle v(E_i), v(E_j) \rangle = -\mathrm{hom}_X(E_i, E_j) + \mathrm{hom}^1_X(E_i, E_j) - \mathrm{hom}_X(E_j, E_i) \geq -2,$$

which implies

$$v(E)^2 = \sum_{i=1}^{n} v(E_i)^2 + 2 \sum_{1 \leq i < j \leq n} \langle v(E_i), v(E_j) \rangle \geq -2n^2 \geq -2c(E)^2.$$

□

A generalization of the argument in [Bay16b, Section 6] implies the following lemma.

Lemma 4.3.2. ***(Brill-Noether wall)*** *Let $\sigma_{(b_0,w_0)}$ be a stability condition such that $b_0 < 0$ and the point $k(b_0, w_0)$ is sufficiently close to the point $pr\big(v(\mathcal{O}_X)\big) = (0, 1) = o'$. Assume an object $E \in \mathcal{D}(X)$ is $\sigma_{(b_0,w_0)}$-semistable and has the same phase as $\mathcal{O}_X$. Then*

$$h^0(X, E) \leq \frac{\chi(E)}{2} + \frac{\sqrt{\big(r(E) - s(E)\big)^2 + c(E)^2(2H^2 + 4)}}{2}, \tag{4.7}$$

where $h^0(X, E) = dim_{\mathbb{C}}\, Hom_X(\mathcal{O}_X, E)$ and $\chi(E) = r(E) + s(E)$ is the Euler characteristic of E.

Proof. By Lemma 4.2.10, part (d) the structure sheaf $\mathcal{O}_X$ is $\sigma_{(0,w)}$-stable for any $w > \sqrt{1/H^2}$. We claim that the structure sheaf $\mathcal{O}_X$ is $\sigma_{(b,w)}$-stable when the point $k(b, w) = pr\big(\ker Z_{(b,w)}\big)$ is sufficiently close to the point $pr\big(v(\mathcal{O}_X)\big) = o'$. Otherwise, $\mathcal{O}_X$ is strictly $\sigma_{(b,w)}$-semistable for a stability condition $\sigma_{(b,w)}$ close to the point o'. Let F be a $\sigma_{(b,w)}$-stable factor of $\mathcal{O}_X$, so $v(F)^2 \geq -2$ and

$$\big|Z_{(b,w)}(F)\big| < \big|Z_{(b,w)}(\mathcal{O}_X)\big| \ll 1.$$

This implies the projection $pr(v(F))$ is sufficiently close to the point o' which is impossible by Lemma 4.2.3 and Remark 4.2.4.

By assumption, the objects E and $\mathcal{O}_X$ have the same phase with respect to the stability condition $\sigma_{(b_0,w_0)}$. Consider the evaluation map

$$\mathrm{ev}\colon \mathrm{Hom}_X(\mathcal{O}_X, E) \otimes \mathcal{O}_X \to E.$$

Since $\mathcal{O}_X$ is $\sigma_{(b_0,w_0)}$-stable, it is a simple object in the abelian category of semistable objects with the same phase as $\mathcal{O}_X$. Therefore, the morphism ev is injective and the cokernel cok(ev) is also $\sigma_{(b_0,w_0)}$-semistable.

Let $\{E_i\}_{i=1}^{i=n}$ be the Jordan-Hölder factors of cok(ev) with respect to $\sigma_{(b_0,w_0)}$. The Mukai vector of any factor is denoted by $w_i := v(E_i) = m_i v(\mathcal{O}_X) + t_i v(E)$ for some $m_i, t_i \in \mathbb{Q}$ where $\sum_{i=1}^n w_i = v(E) - h^0(E)\, v(\mathcal{O}_X)$.

If we deform the stability condition $\sigma_{(b_0,w_0)}$ along the line segment that connects $k(b_0, w_0)$ to the point o', the objects E_i remain stable and of the same phase as E and $\mathcal{O}_X$. Therefore the Jordan-Hölder filtration of cok(ev) does not change and

$$\lim_{k(b,w)\to(0^-,1)} \mathrm{Im}[Z_{(b,w)}(E_i)] = \lim_{b\to 0^-} \left[t_i\big(c(E) - br(E)\big) + m_i(-b)\right] \geq 0.$$

This implies $t_i \geq 0$ and if $t_i = 0$, then $E_i \cong \mathcal{O}_X$. Since $t_i c(E) \in \mathbb{Z}$ and $\sum_{i=1}^n t_i = 1$, the maximum number of factors with $t_i \neq 0$ is equal to $c(E)$. By reordering of the factors, we can assume $E_i \cong \mathcal{O}_X$ for $1 \leq i \leq i_0$ and other factors satisfy $t_i \neq 0$. Therefore,

$$v(E) - \big(h^0(X,E) + i_0\big)v(\mathcal{O}_X) = \sum_{i=i_0+1}^{n} w_i$$

where $0 \leq n - i_0 \leq c(E)$. Since $\langle w_i, w_j \rangle \geq -2$ for $1 \leq i, j \leq n$, the same argument as in Lemma 4.3.1 implies that

$$\big(v(E) - (h^0(X,E) + i_0)v(\mathcal{O}_X)\big)^2 = \left(\sum_{i=i_0+1}^{n} w_i\right)^2 \geq -2c(E)^2.$$

Now solving the quadratic equation

$$f(x) = \big(v(E) - xv(\mathcal{O}_X)\big)^2 + 2c(E)^2 = -2x^2 + 2x\,\chi(E) + v(E)^2 + 2c(E)^2 = 0$$

shows that

$$h^0(X,E) \leq h^0(X,E) + i_0 \leq \frac{\chi(E)}{2} + \frac{\sqrt{\big(r(E) - s(E)\big)^2 + c(E)^2(2H^2+4)}}{2}.$$

□

Definition 4.3.3. Given a stability condition $\sigma_{(b,w)}$ and an object $E \in \mathcal{A}(b)$, the Harder-Narasimhan polygon $\mathrm{HN}^{\sigma_{(b,w)}}(E)$ is the convex hull of the points $Z_{(b,w)}(E')$ for all subobjects $E' \subset E$ of E.

If the Harder-Narasimhan filtration of E is the sequence

$$0 = \tilde{E}_0 \subset \tilde{E}_1 \subset .... \subset \tilde{E}_{n-1} \subset \tilde{E}_n = E,$$

then the points $\{p_i = Z_{(b,w)}(\tilde{E}_i)\}_{i=0}^{i=n}$ are the extremal points of the polygon $\mathrm{HN}^{\sigma_{(b,w)}}(E)$ on the left side of the line segment $\overline{oZ_{(b,w)}(E)}$, see Figure 4.5.

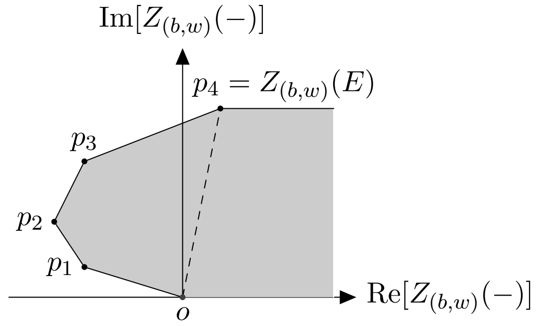

Figure 4.5: The HN polygon is in the grey area.

We define the following non-standard norm on $\mathbb{C}$:

$$\|x + iy\| = \sqrt{x^2 + (2H^2 + 4)y^2}.$$

The function $\overline{Z}\colon K(X) \to \mathbb{C}$ is defined as $\overline{Z}(E) = Z_{(0,\sqrt{2/H^2})}(E) = r(E) - s(E) + i\, c(E)$. The next proposition shows that we can bound the number of global sections of an object in $\mathcal{A}(0)$ via the length of Harder-Narasimhan polygon at some limit point.

Proposition 4.3.4. *Let $E \in \mathcal{A}(0)$ be an object which has no subobject $F \subset E$ with $ch_1(F) = 0$.*

(a) There exists $w^* > \sqrt{2/H^2}$ such that the Harder-Narasimhan filtration of E is a fixed sequence

$$0 = \tilde{E}_0 \subset \tilde{E}_1 \subset .... \subset \tilde{E}_{n-1} \subset \tilde{E}_n = E,$$

for all stability conditions $\sigma_{(0,w)}$ where $\sqrt{2/H^2} < w < w^*$.

(b) Let $p_i \coloneqq \overline{Z}(\tilde{E}_i)$ for $0 \leq i \leq n$, then

$$h^0(X, E) \leq \frac{\chi(E)}{2} + \frac{1}{2}\sum_{i=1}^{n} \|\overline{p_i p_{i-1}}\|.$$

Proof. We first show that there exists $w_1 > \sqrt{2/H^2}$ such that the semistable factor $\tilde{E}_1$ is fixed for the stability conditions of form $\sigma_{(0,w)}$ where $\sqrt{2/H^2} < w < w_1$.

Let $\sigma_{(0,w)}$ be a stability condition such that $\sqrt{2/H^2} < w < \sqrt{4/H^2} \coloneqq w_0$ and $v_1 = (r_1, c_1 H, s_1)$ be a possible class of the semistable factor $\tilde{E}_1$. Lemma 4.3.1 implies that

$$r_1 s_1 \leq c_1^2\left(\frac{H^2}{2} + 1\right) \leq c(E)^2\left(\frac{H^2}{2} + 1\right). \tag{4.8}$$

Since $\phi_{(0,w)}(v(E)) \leq \phi_{(0,w)}(v_1)$, we have

$$\mathrm{Re}[Z_{(0,w)}(v_1)] \leq \max\{\mathrm{Re}[Z_{(0,w)}\big(v(E)\big)], 0\}.$$

If $r_1 \geq 0$ and $s_1 \leq 0$, then we have

$$\max\{r_1, -s_1\} \leq \frac{r_1 H^2 w^2}{2} - s_1 = \mathrm{Re}[Z_{(0,w)}(v_1)] \leq \max\{\mathrm{Re}[Z_{(0,w)}\big(v(E)\big)], 0\}. \tag{4.9}$$

If $r_1 < 0$ and $s_1 > 0$, the existence of HN filtration for the object E at $\sigma_{(0,w_0)}$ implies that there exists a real number M_0 such that

$$M_0 \le \mathrm{Re}[Z_{(0,w_0)}(v_1)] \le r_1 - s_1. \tag{4.10}$$

Inequalities (4.8), (4.9) and (4.10) imply that there are only finitely many possible classes v_1. Thus there exists $w_1 > \sqrt{2/H^2}$ such that the semistable factor $\tilde{E}_1$ is fixed with respect to $\sigma_{(0,w)}$ where $\sqrt{2/H^2} < w < w_1$.

Continuing this argument by induction, one shows that there is a number w_i such that $\sqrt{2/H^2} < w_i < w_{i-1}$ and the semistable factor $E_i = \tilde{E}_i/\tilde{E}_{i-1}$ which is the semistable subobject of $E/\tilde{E}_{i-1}$ with the maximum phase, is fixed for the stability conditions $\sigma_{(0,w)}$ where $\sqrt{2/H^2} < w < w_i$. Note that $\phi^+_{(0,w)}(E/\tilde{E}_i) < \phi^+_{(0,w)}(E) < 1$, thus $0 < \mathrm{Im}[Z_{(0,w)}(E_i)] \le c(E)$ and the length of the HN filtration of E is at most $c(E)$. This completes the proof of (a).

Since $c(E_i) \neq 0$ for $1 \le i \le n$, the point $pr(v(E_i))$ is not on the y-axis. Proposition 4.2.6, part (d) implies that the line segment $\overline{oo'}$ is not a wall for the semistable factor E_i. Thus the stability conditions $\sigma_{(0,w)}$ for $\sqrt{2/H^2} < w < w^*$ are all inside one chamber for E_i. We denote the point $k(0,w^*)$ by o^*. If $s(E_i) \neq 0$, then define V as a cone in $\mathbb{R}^2$ with two rays $\overline{pr_i o'}$ and $\overline{pr_i o^*}$ where $pr_i \coloneqq pr(v(E_i))$. If $s(E_i) = 0$, then V is the area between two parallel lines of slope $r(E_i)/c(E_i)$ which pass through the points o' and o^*, see Figure 4.6.

Lemma 4.2.3 implies that there are no projection of roots other than $pr(v(\mathcal{O}_X))$ inside the rectangle with vertices $a_1 = \big(1/2, H^2/8\big)$, $a_2 = \big(1/2, 3/2\big)$, $a_3 = \big(-1/2, 3/2\big)$ and $a_4 = \big(-1/2, H^2/8\big)$. Let V' be the intersection of V and the rectangle $a_1a_2a_3a_4$, see the dashed area in Figure 4.6. The structure of the wall and chamber decomposition implies that E_i is semistable with respect to the stability conditions in V'. In particular, it is $\sigma_i \coloneqq \sigma_{(b_i,w_i)}$-semistable where σ_i is on the top boundary of V', i.e., the associated point $k(b_i,w_i)$ is on the top boundary of V'.

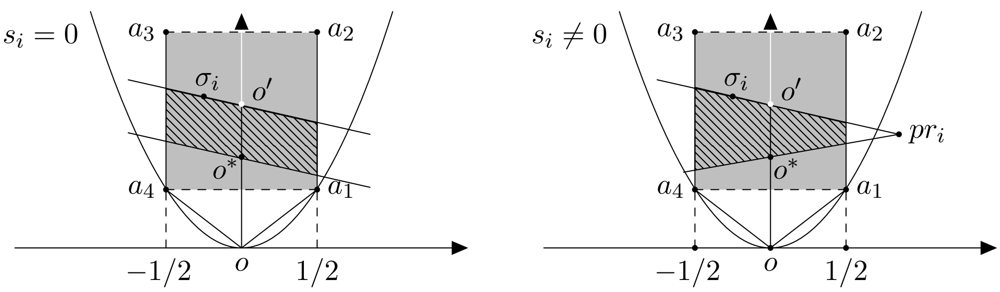

Figure 4.6: The object E_i remains semistable when we go to σ_i

We may assume $-1 \ll b_i < 0$, so $\mathrm{Im}[Z_{(b_i,w_i)}(E_i)] = c(E_i) - r(E_i)b_i > 0$ and $E_i \in \mathcal{A}(b_i)$. By Lemma 3.2.1, the objects E_i and $\mathcal{O}_X$ are σ_i-semistable of the same phase and Lemma 4.3.2 gives

$$h^0(E_i) \le \frac{r(E_i) + s(E_i) + \sqrt{\big(r(E_i) - s(E_i)\big)^2 + c(E_i)^2(2H^2+4)}}{2} = \frac{r(E_i)+s(E_i)}{2} + \frac{\|\overline{p_ip_{i-1}}\|}{2}.$$

Thus,

$$h^0(E) \le \sum_{i=1}^{n} h^0(E_i) \le \frac{\chi(E)}{2} + \frac{1}{2}\sum_{i=1}^{n} \|\overline{p_ip_{i-1}}\|.$$

□

Remark 4.3.5. Let P_E be the convex hull of the points $\{p_0, p_1, ..., p_n\}$ as defined in Proposition 4.3.4, part (b). We think of P_E as the Harder-Narasimhan polygon of E on the left at the limit point.

We finish this section by stating two useful inequalities which are the result of deformation of stability conditions.

Lemma 4.3.6. *Let $\sigma_{(b_0,w_0)}$ be a stability condition and $E \in \mathcal{A}(b_0)$ be a $\sigma_{(b_0,w_0)}$-semistable object with $c(E) \neq 0$. Assume L is a line which passes through the point $pr(v(E))$ if $s(E) \neq 0$ or it has a slope of $r(E)/c(E)$ if $s(E) = 0$. Assume $q_1 = (x_1, y_1)$ and $q_2 = (x_2, y_2)$ are two points on the line L where $y_1 y_2 \neq 0$ and $x_1/y_1 \leq x_2/y_2$. Assume also the point $k(b_0, w_0)$ is on the open line segment $(\overline{q_1 q_2})$. If every point on the open line segment $(\overline{q_1 q_2})$ is in correspondence to a stability condition, i.e., if $(\overline{q_1 q_2}) \subset V(X)$, then*

$$\mu_H^+\big(H^{-1}(E)\big) \leq \frac{x_1}{y_1} \quad \text{and} \quad \frac{x_2}{y_2} \leq \mu_H^-\big(H^0(E)\big). \tag{4.11}$$

Proof. The construction of the walls and Proposition 4.2.6 imply that E is $\sigma_{(b,w)}$-semistable and it is in the heart $\mathcal{A}(b)$ whenever the point $k(b,w)$ is on the open line segment $(\overline{q_1 q_2})$. Therefore,

$$\mu_H^+\big(H^{-1}(E)\big) \leq b < \mu_H^-\big(H^0(E)\big).$$

If $k(b_i, w_i) = q_i$, then $b_i = x_i/y_i$. Thus the stability conditions close to the points q_1 or q_2 give the inequalities (4.11). □

4.4 The Brill-Noether loci in the case (A)

In this section, we first show that the morphism $\psi\colon M_{X,H}(\bar{v}) \to$BN described in (4.1) is well-defined in case (A). Then we consider a slope semistable rank r-vector bundle F on the curve C of degree $2rs$ and describe the location of the wall that bounds the Gieseker chamber for the push-forward of F. Finally, in Proposition 4.4.4, we show that if the number of global sections of F is high enough, then it must be the restriction of a vector bundle on the surface.

We assume throughout Section 4.4 that X is a K3 surface with $\mathrm{Pic}(X) = \mathbb{Z}.H$ and $H^2 = 2rs$ for some $r \geq 2$ and $s \geq \max\{r, 5\}$. We also assume C is a curve in the linear system $|H|$ and $i\colon C \hookrightarrow X$ is the embedding of the curve C into the surface. Let $M_C(r, 2rs)$ be the moduli space of slope semistable rank r-vector bundles on the curve C of degree $2rs$. The push-forward of any vector bundle $F \in M_C(r, 2rs)$ to the surface X has Mukai vector

$$v \coloneqq v(i_*F) = (0, rH, 2rs - r^2 s).$$

Lemma 4.2.10, part (a) implies that the push-forward i_*F is $\sigma_{(b,w)}$-semistable where $w \gg 0$. The chamber which contains these stability conditions is called the *Gieseker chamber*. Note that the corresponding point $k(b,w)$ is close to the origin.

Let $M_{X,H}(\bar{v})$ be the moduli space of H-Gieseker semistable sheaves on the surface X with Mukai vector $\bar{v} = (r, H, s)$. Since $\bar{v}^2 = 0$ and $\bar{v}$ is primitive, the moduli space $M_{X,H}(\bar{v})$ is a smooth projective K3 surface [Huy16, Proposition 2.5 and Corollary 3.5].

Any coherent sheaf $E \in M_{X,H}(\bar{v})$ is a μ_H-stable locally free sheaf [HL10, Remark 6.1.9]. Note that $E(-H)$ is also μ_H-stable. Let

$$q := pr(\bar{v}) = \left(\frac{1}{s}, \frac{r}{s}\right) \quad \text{and} \quad p := pr(v - \bar{v}) = pr\big(v(E(-H))\big) = \left(\frac{-1}{s(r-1)}, \frac{r}{s(r-1)^2}\right).$$

We also denote by $\tilde{o}$ the point at which the line segments $\overline{pq}$ and $\overline{o'o}$ intersect, where $o' = pr(v(\mathcal{O}_X))$ as before. Define the object $W_E \in \mathcal{D}(X)$ as the cone of the evaluation map:

$$\mathcal{O}_X^{h^0(X,E)} \xrightarrow{\mathrm{ev}_E} E \to W_E. \tag{4.12}$$

Denote the point $\big(-1/r, s/r\big)$ by q'. Lemma 4.2.5 for $m = r$, $n = s(r-1)$ and $\epsilon = 1$ implies that there is no projection of roots in the grey area and on the open line segment $(\overline{et})$ in Figure 4.7, where $e = q_{-m,-n,-\epsilon}$ and $t = q'_{-m,-n,-\epsilon}$. As before, we denote by γ_n the point on the parabola $y = rsx^2$ with the x-coordinate $1/n$.

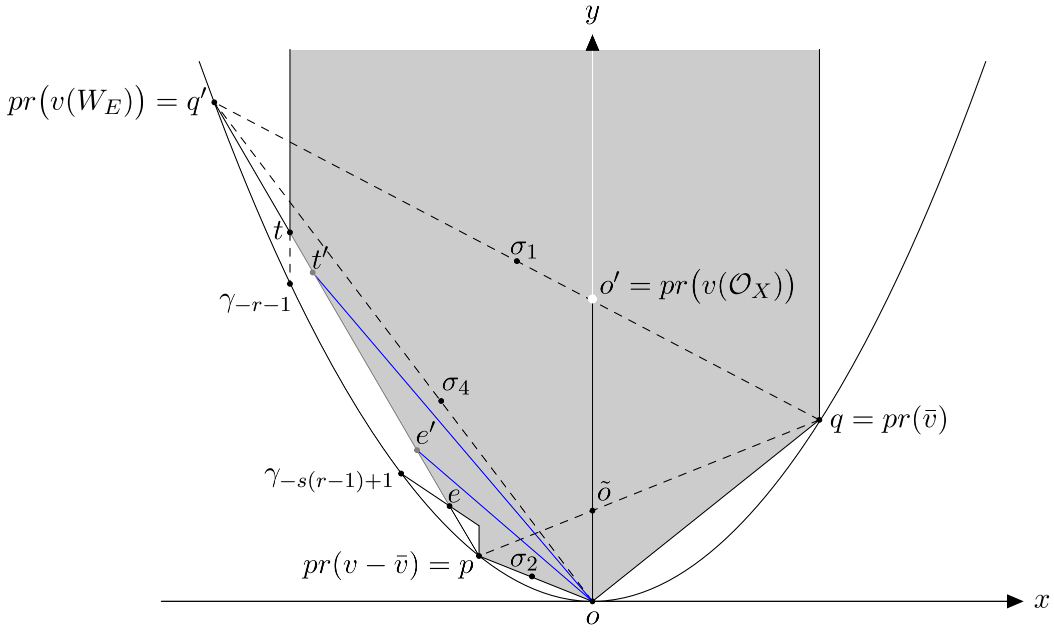

Figure 4.7: No projection of roots in the grey area

Proposition 4.4.1. *Let $E \in M_{X,H}(\bar{v})$ be a μ_H-stable vector bundle on the surface X.*

(a) The restriction $E|_C$ is a slope stable vector bundle on the curve C and $h^0(C, E|_C) = r + s$. In particular, the morphism ψ described in (4.1) is well-defined in case (A).

(b) $Hom_X\big(E, E(-H)[1]\big) = 0$.

(c) The object W_E is of the form $W_E = E'[1]$ where E' is a μ_H-stable locally free sheaf on X and $Hom_X\big(E', E(-H)[1]\big) = 0$.

Proof. By Lemma 4.2.10, part (a), the coherent sheaf E is $\sigma_{(0,w)}$-stable where $w \gg 0$. Lemma 4.2.7 implies that there is no wall for E intersecting the line segment $\overline{oo'}$. If the stability condition $\sigma_1 := \sigma_{(b_1,w_1)}$ is on the line segment $\overline{q'o'}$ (i.e. the corresponding point $k(b_1, w_1)$ is on $\overline{q'o'}$), then E is σ_1-semistable and has the same phase as the structure sheaf $\mathcal{O}_X$. Then Lemma 4.3.2 shows that $h^0(X, E) \leq r + s$. Moreover, E has positive slope, so $\mathrm{Hom}_X(E, \mathcal{O}_X) = 0$ and

$$\chi(E) = r + s = h^0(X, E) - h^1(X, E).$$

Therefore $h^0(X, E) = r + s$ and the object W_E has Mukai vector $v(W_E) = (-s, H, -r)$ with projection $pr\big(v(W_E)\big) = q'$.

Lemma 4.2.10, part (d) implies that $E(-H)[1]$ with Mukai vector $\big(-r, (r-1)H, -s(r-1)^2\big)$ is $\sigma_2 := \sigma_{(b_2, w_2)}$-stable where $b_2 = -(r-1)/r$ and $w_2 \gg 0$. Note that Lemma 4.2.3 ensures that such a stability condition exists. Let e' be the point that the line segment $\overline{q'p}$ intersects the line given by the equation $x = b_3 y$, where

$$b_3 = -\frac{r-2}{r-1} \quad \text{if } r > 2 \qquad \text{or} \qquad b_3 = -\frac{1}{3} \quad \text{if } r = 2.$$

If $s \geq \max\{r, 5\}$, then

$$-\frac{s(r-1)-1}{rs} < b_3 < -\frac{r+1}{rs}.$$

Thus the line segment $\overline{oe'}$ is located between two lines $\overline{o\gamma_{-r-1}}$ and $\overline{o\gamma_{-s(r-1)+1}}$ and it is on the grey area with no projection of roots. By Lemma 4.2.7, there is no wall for $E(-H)[1]$ intersecting the closed line segment $[\overline{oe'}]$. Therefore, $E(-H)$ is stable with respect to the stability condition at the point e'. In particular, this implies the structure sheaf $\mathcal{O}_X$ does not make a wall for $E(-H)[1]$, so

$$\mathrm{Hom}_X(\mathcal{O}_X, E(-H)[1]) = 0.$$

To prove (b), we consider the stability condition $\tilde{\sigma} := \sigma_{(0,\tilde{w})}$ where $k(0, \tilde{w}) = \tilde{o}$, see Figure 4.7. The objects E and $E(-H)[1]$ are $\tilde{\sigma}$-stable of the same phase, so $\mathrm{Hom}_X(E, E(-H)[1]) = 0$. Moreover, $i_*E|_C$ is the extension of these two objects in $\mathcal{A}(0)$,

$$E \hookrightarrow i_*E|_C \twoheadrightarrow E(-H)[1].$$

Hence $i_*E|_C$ is $\tilde{\sigma}$-semistable. One can show $\phi_{(0,w)}(E) < \phi_{(0,w)}(i_*E|_C)$ for $w > \tilde{w}$, so the uniqueness of the JH filtration implies that $i_*E|_C$ is $\sigma_{(0,w)}$-stable and Lemma 4.2.10, part (c) shows that $E|_C$ is slope-stable. Moreover, $E(-H)$ has negative slope, so

$$\mathrm{Hom}_X\big(\mathcal{O}_X, E(-H)\big) = 0.$$

This implies $h^0(C, E|_C) = h^0(X, E) = r + s$, which completes the proof of (a).

The sheaves E and $\mathcal{O}_X$ are σ_1-semistable of the same phase. Applying the same argument as in Lemma 4.3.2, one can show that the object W_E is the cokernel of the evaluation map in the abelian category of semistable objects with the same phase as O_X; hence it is σ_1-semistable. We claim that W_E is σ_1-stable. Indeed, if there exists a subobject $E_1 \subset W_E$ with the same phase as W_E, then $v(E_1) = t_1 v(E) + s_1 v(\mathcal{O}_X)$ where $0 \leq t_1 \leq 1$. Since $t_1 c(E) = t_1 \in \mathbb{Z}$, we have $t_1 = 0, 1$. Therefore, $\mathcal{O}_X$ is a subobject or a quotient of W_E. But, $\mathrm{Hom}(\mathcal{O}_X, W_E) = 0$ and since $\mathrm{Hom}(E, \mathcal{O}_X) = 0$, we have $\mathrm{Hom}(W_E, \mathcal{O}_X) = 0$, a contradiction.

Lemma 4.2.7 shows that there is no wall for W_E intersecting the open line segment $(\overline{oo'})$. Therefore, it is $\sigma_{(0,w)}$-stable where $w \gg 0$. By [MS16, Lemma 6.18], $H^0(W_E)$ is zero or a skyscraper sheaf and $H^{-1}(W_E)$ is a μ_H-stable sheaf. If $H^0(W_E) \neq 0$, then for some $k > 0$, we have

$$v\big(H^{-1}(W_E)\big)^2 = (s, -H, r+k)^2 = -2sk < -2,$$

a contradiction. Therefore, $W_E = E'[1]$ for a locally free sheaf E' on X and it is $\sigma_4 := \sigma_{(b_4,w_4)}$-stable where $b_4 = -1/s$ and $w_4 \gg 0$, by Lemma 4.2.10, part (d).

Let t' be the point that the line segment $\overline{q'p}$ intersects the line given by the equation $y = x(-s+1)$. Then the x-coordinate of the point t' is equal to $-1/(2r-1)$ which is bigger than $-1/(r+1)$ if $r > 2$ and $t' = t$ if $r = 2$. We claim that for $r = 2$ the point $t = \big(-1/3, (s-1)/3\big)$ cannot be the projection of a root. Indeed, if there exists a root $\delta = (\tilde{r}, \tilde{c}H, \tilde{s})$ with $pr(\delta) = t$, then

$$\frac{\tilde{c}}{\tilde{s}} = \frac{-1}{3} \quad \text{and} \quad \frac{\tilde{r}}{\tilde{s}} = \frac{s-1}{3}.$$

This implies $|\tilde{s}| \geq 3$. Since $\delta^2 = -2$, we have $\tilde{s}^2(s-3) = 9$ which is impossible for $s \geq 5$. By Lemma 4.2.7, there is no wall for E' intersecting the closed line segment $[\overline{ot'}]$. Thus, E' is stable with respect to the stability condition at the point e' and it has the same phase as $E(-H)[1]$, so there is no non-trivial homomorphism between them which finishes the proof of (c). □

4.4.1 The first wall

Lemma 4.2.10, part (a) implies that the push-forward of any vector bundle $F \in M_C(r, 2rs)$ is $\sigma_0 := \sigma_{(b_0,w_0)}$-semistable where σ_0 is in the Gieseker chamber for i_*F, which means w_0 is large enough and b_0 is arbitrary. By Proposition 4.2.6, part (d), any wall for i_*F is part of a line which goes through the point $p' := pr\big(v(i_*F)\big)$ if $r > 2$ or it is a horizontal line segment if $r = 2$. The next proposition describes the location of the wall that bounds the Gieseker chamber for i_*F. Recall that $q = pr(\bar{v})$ and $p = pr(v - \bar{v})$.

Proposition 4.4.2. *Given a vector bundle $F \in M_C(r, 2rs)$, the wall that bounds the Gieseker chamber for i_*F is not below the line segment $\overline{pq}$ and it coincides with the line segment $\overline{pq}$ if and only if F is the restriction of a vector bundle $E \in M_{X,H}(\bar{v})$ to the curve C.*

Proof. Assume that the wall $\mathcal{W}_{i_*F}$ that bounds the Gieseker chamber for i_*F, is below or on the line segment $\overline{pq}$, see Figure 4.8 for $r > 2$.

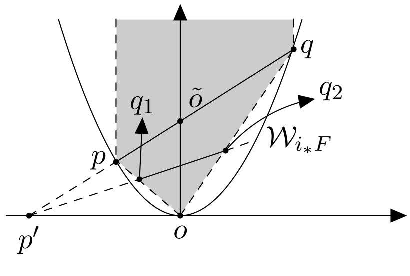

Figure 4.8: The first wall $\mathcal{W}_{i_*F}$

Suppose the stability condition $\sigma_{(0,w^*)}$ is on the wall $\mathcal{W}_{i_*F}$. Then there is a destabilising sequence

$$F_1 \hookrightarrow i_*F \twoheadrightarrow F_2$$

of objects in $\mathcal{A}(0)$ such that F_1 and F_2 are $\sigma_{(0,w^*)}$-semistable of the same phase as i_*F and $\phi_{(0,w)}(F_1) > \phi_{(0,w)}(i_*F)$ for $w < w^*$. Taking cohomology gives a long exact sequence of sheaves

$$0 \to H^{-1}(F_2) \to F_1 \xrightarrow{d_0} i_*F \xrightarrow{d_1} H^0(F_2) \to 0. \tag{4.13}$$

Let $v(F_1) = \big(r', c'H, s'\big)$ and $v\big(H^0(F_2)\big) = \big(0, c''H, s''\big)$. If $r' = 0$, then since F_1 and i_*F have the same phase with respect to $\sigma_{(0,w^*)}$, we have $v(F_1) = k.v(i_*F)$ for some $k \in \mathbb{R}$. This implies F_1 and i_*F have the same phase with respect to all stability conditions in $V(X)$ and so F_1 cannot make a wall for i_*F, hence $r' > 0$.

Let $T(F_1)$ be the maximal torsion subsheaf of F_1 and $v\big(T(F_1)\big) = (0, \tilde{r}H, \tilde{s})$. Let $F_1/T(F_1)$ be the torsion free part of F_1. The torsion sheaf $T(F_1)$ is also a subsheaf of i_*F, thus

$$\operatorname{rank}(i^*F_1) \leq \operatorname{rank}\big(i^*T(F_1)\big) + \operatorname{rank}\big(i^*\big(F_1/T(F_1)\big)\big) = \tilde{r} + r'.$$

Note that i^* is always underived. The surjection $F_1 \twoheadrightarrow \ker d_1$ factors through $F_1 \twoheadrightarrow i_*i^*F_1 \twoheadrightarrow \ker d_1$. Thus,

$$\operatorname{rank}\big(i^* \ker d_1\big) \leq \operatorname{rank}(i^*F_1) \quad \Rightarrow \quad r - c'' \leq r' + \tilde{r}. \tag{4.14}$$

Assume q_1 and q_2 are the points of intersection of the wall $\mathcal{W}_{i_*F}$ with the line segments $\overline{op}$ and $\overline{oq}$, respectively. Then Lemma 4.3.6 implies that

$$\frac{1}{r} \leq \mu_H^-(F_1) = \mu_H^-\big(F_1/T(F_1)\big) \quad \text{and} \quad \mu_H^+\big(H^{-1}(F_2)\big) \leq \frac{1-r}{r}.$$

Therefore,

$$\frac{r - c'' - \tilde{r}}{r'} = \mu_H\big(F_1/T(F_1)\big) - \mu_H\big(H^{-1}(F_2)\big) \geq$$
$$\mu_H^-\big(F_1/T(F_1)\big) - \mu_H^+\big(H^{-1}(F_2)\big) \geq \frac{1}{r} - \frac{1-r}{r} = 1.$$

Combined with the inequality (4.14), this is only possible if all these inequalities are equalities, i.e. $r' = r - c'' - \tilde{r}$,

$$\mu_H^+\big(H^{-1}(F_2)\big) = \mu_H\big(H^{-1}(F_2)\big)$$

and

$$\mu_H\big(F_1/T(F_1)\big) = \mu_H^-\big(F_1/T(F_1)\big) = \frac{1}{r} = \frac{c' - \tilde{r}}{r - c'' - \tilde{r}}.$$

Therefore, $c'' = \tilde{r} = 0$ and $c' = 1$, which implies that F_1 is a torsion-free sheaf with Mukai vector $v(F_1) = (r, H, s')$. Note that $T(F_1)$ cannot be a skyscraper sheaf because F_1 is $\sigma_{(0,w^*)}$-stable. Moreover, the above equalities imply that F_1 and $H^{-1}(F_2)$ are μ_H-stable sheaves. Since the point $pr\big(v(F_1)\big)$ lies on the extension of $\mathcal{W}_{i_*F}$, we have $pr(v(F_1)) = \big(1/s', r/s'\big) = q_2$.

If the wall $\mathcal{W}_{i_*F}$ is below the line segment $\overline{pq}$, then $s' > s$, but $\sigma_{(0,w^*)}$-stability of F_1 gives

$$v(F_1)^2 = 2r(s - s') \geq -2 \quad \Rightarrow \quad s' \leq s,$$

this is a contradiction.

If the wall $\mathcal{W}_{i_*F}$ coincides with the line segment $\overline{pq}$, then $v(F_1) = \big(r, H, s\big)$. The non-zero morphism d_0 in the long exact sequence (4.13) factors via the morphism $d_0'\colon i_*F_1|_C \to i_*F$. The objects $i_*F_1|_C$ and i_*F have the same Mukai vector and so have the same phase. Proposition 4.4.1 implies that $i_*F_1|_C$ is H-Gieseker stable. Hence the morphism d_0' is injective and so $F_1|_C \cong F$. □

Now instead of checking the possible walls above the line segment $\overline{pq}$, we consider the stability conditions of form $\sigma_{(0,w)}$ which are close to the point $(0,1)$ and examine the Harder-Narasimhan filtrations. Given a semistable vector bundle $F \in M_C(r, 2rs)$, the $\sigma_{(0,w)}$-semistability of i_*F for $w \gg 0$ gives $\phi^+_{(0,w)}(i_*F) < 1$. Proposition 4.3.4 implies that there is a positive real number $w' > 0$ such that for every stability condition $\sigma_{(0,w)}$ where $\sqrt{1/(rs)} < w < w'$, the HN filtration of i_*F is a fixed sequence

$$0 = \tilde{E}_0 \subset \tilde{E}_1 \subset .... \subset \tilde{E}_{n-1} \subset \tilde{E}_n = i_*F$$

with the semistable factors $E_i = \tilde{E}_i/\tilde{E}_{i-1}$ for $1 \leq i \leq n$. Recall that the stability function $\overline{Z}\colon K(X) \to \mathbb{C}$ is defined as $\overline{Z}(E) = Z_{\left(0,\sqrt{1/rs}\right)}(E)$. Let P_F be the polygon with the vertices $\{p_i\}_{i=0}^{i=n}$ where $p_i = \overline{Z}(\tilde{E}_i)$ and the triangle T has vertices $g_1 \coloneqq \overline{Z}(\bar{v}) = r - s + i$, $g_2 \coloneqq \overline{Z}(i_*F) = r^2s - 2rs + i\,r$ and the origin.

Lemma 4.4.3. *The polygon P_F for any vector bundle $F \in M_C(r, 2rs)$ is contained in the triangle $T = \triangle og_1g_2$ and they coincide if and only if the bundle F is the restriction of a vector bundle $E \in M_{X,H}(\bar{v})$ to the curve C, see Figure 4.9.*

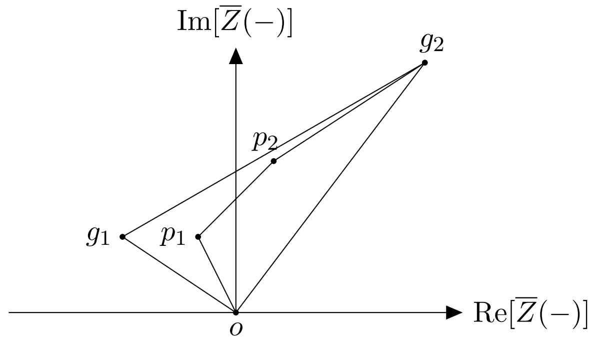

Figure 4.9: The polygon P_F is inside the triangle T

Proof. For the first statement, it suffices to show that

$$\phi_{(0,\sqrt{1/rs})}\big(v(E_1)\big) \leq \phi_{(0,\sqrt{1/rs})}(\bar{v}) \quad \text{and} \quad \phi_{(0,\sqrt{1/rs})}\big(v(E_n)\big) \geq \phi_{(0,\sqrt{1/rs})}(v - \bar{v}). \tag{4.15}$$

Let $v(E_1) = (r_1, c_1H, s_1)$. Since $\phi^+_{(0,w)}(i_*F) < 1$, we have $0 < c_1 \leq r$. Assume for a contradiction that

$$\phi_{(0,\sqrt{1/rs})}(E_1) > \phi_{(0,\sqrt{1/rs})}(\bar{v}) \quad \Rightarrow \quad \frac{s_1}{c_1} - \frac{r_1}{c_1} > s - r. \tag{4.16}$$

Therefore, the point $q_1 \coloneqq (r_1/c_1, s_1/c_1)$ is above the line L_1 given by the equation $y - x = s - r$. Proposition 4.4.2 implies that F is $\sigma_{(0,\tilde{w})}$-semistable, thus

$$\phi_{(0,\tilde{w})}\big(E_1\big) \leq \phi_{(0,\tilde{w})}(\bar{v}) \quad \Rightarrow \quad \frac{s_1}{c_1} - \frac{r_1}{c_1}(rs\tilde{w}^2) \leq s - r(rs\tilde{w}^2). \tag{4.17}$$

This shows q_1 is below or on the line L_2 given by the equation $y - x(rs\tilde{w}^2) = s - r^2 s\tilde{w}^2$, see Figure 4.10. Since the point of intersection of the lines L_1 and L_2 is (r, s), we must have

$$r < \frac{r_1}{c_1} \quad \Rightarrow \quad r \leq \frac{r_1}{c_1} - \frac{1}{c_1} \leq \frac{r_1}{c_1} - \frac{1}{r}.$$

Therefore, the point q_1 is in the dashed area in Figure 4.10. The point on the line L_1 with the first coordinate $r + 1/r$, which is denoted by q', has the second coordinate $s + 1/r$, so q' is above the hyperbola with equation $xy = rs + 1$. This implies the point q_1 is also above the hyperbola. Thus,

$$rs + 1 < \frac{r_1 s_1}{c_1^2}$$

which is a contradiction by Lemma 4.3.1.

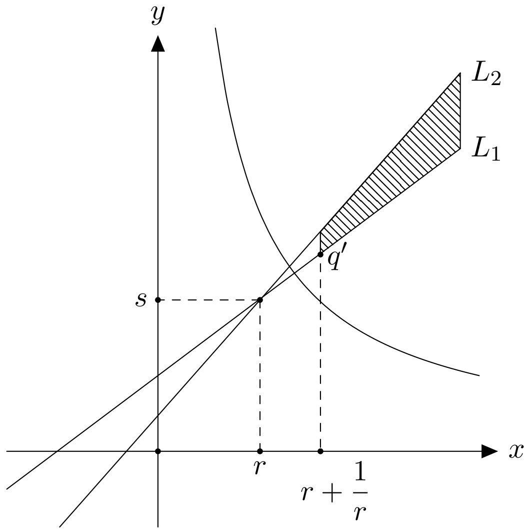

Figure 4.10: The point q_1 is on the dashed area

Similarly, if the semistable factor E_n with Mukai vector $v(E_n) = (r_n, c_n H, s_n)$, does not satisfy the inequality (4.15), then the point $q_n \coloneqq (r_n/c_n, s_n/c_n)$ is below the line L_1' by the equation $y = x - s(r-1) + r/(r-1)$ and is above or on the line L_2' with the equation $y = x(rs\tilde{w}^2) - s(r-1) + r^2 s\tilde{w}^2/(r-1)$. Since the point of intersection of these two lines is $\big(-r/(r-1), -s(r-1)\big)$, we have

$$\frac{r_n}{c_n} < \frac{-r}{r-1} \quad \Rightarrow \quad \frac{r_n}{c_n} \leq \frac{-r}{r-1} - \frac{1}{c_n(r-1)} \leq \frac{-r}{r-1} - \frac{1}{(r-1)^2}.$$

Then the same argument as above leads to a contradiction if $s \geq r$.

Moreover, if F is the restriction of a vector bundle $E \in M_{X,H}(\bar{v})$, then Proposition 4.4.1 implies that the HN factors of $i_* E|_C$ with respect to the stability conditions close to the point $(0, 1)$, are E and $E(-H)[1]$. Therefore, the polygon P_F coincides with T.

Conversely, given a vector bundle $F \in M_C(r, 2rs)$ such that $P_F = T$, then $g_1 = p_1$ and $v(E_1) = (r+k, H, s+k)$. The point $q_1 = (r+k, s+k)$ is on the line L_1. Since q_1 is in the dashed area in Figure 4.10, we have $k \geq 0$. But, $v(E_1)^2 = -2k(r+k+s) \geq -2$ which gives $k = 0$ and Proposition 4.4.2 implies that F is the restriction of the vector bundle $E_1 \in M_{X,H}(\bar{v})$. $\square$

4.4.2 The maximum number of global sections

The next proposition shows that any vector bundle $F \in M_C(r, 2rs)$ with high enough number of global sections is the restriction of a vector bundle on the surface.

Proposition 4.4.4. *Let F be a slope-semistable rank r-vector bundle on the curve C of degree $2rs$, where $s \geq \max\{5, r\}$. If $h^0(F) \geq r+s$, then F is the restriction of a vector bundle $E \in M_{X,H}(\bar{v})$ to the curve C. In particular, the morphism $\psi\colon M_{X,H}(\bar{v}) \to BN$, which sends a vector bundle to its restriction, is bijective in case (A).*

Proof. If the vector bundle $F \in T$ is the restriction of a vector bundle $E \in M_{X,H}(\bar{v})$, then E is a Harder-Narasimhan factor of i_*F with respect to $\sigma_{(0,w)}$ where $\sqrt{1/rs} < w < w_1$. Thus the uniqueness of the Harder-Narasimhan filtration implies that ψ is injective.

For the surjectivity part, Lemma 4.4.3 implies that we only need to show that the polygon P_F coincides with the triangle $T = \triangle\, og_1g_2$. Assume for a contradiction that P_F is strictly inside T. Since the vertices of P_F are Gaussian integers, P_F must be contained in the polygon $og_1'g_2'g_2$, where $g_1' = r-s+1+i$, $g_2' = s(r-2)+r-r/(r-1)+2i$, see Figure 4.11.

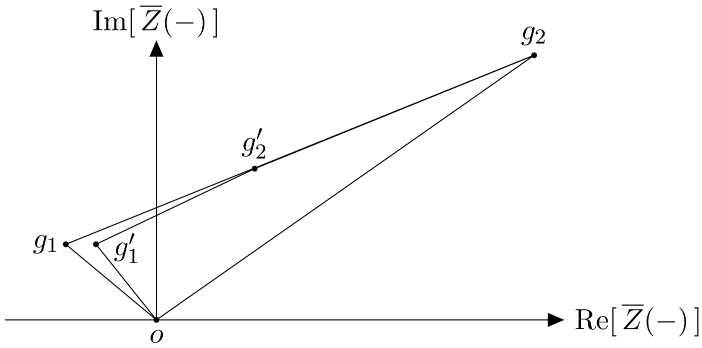

Figure 4.11: The polygon P_F is inside the polygon $og_1'g_2'g_2$

The convexity of the polygon P_F and the polygon $og_1'g_2'g_2$ gives $\sum_{i=1}^{n} \|\overline{p_ip_{i-1}}\| \leq \|\overline{og_1'}\| + \|\overline{g_1'g_2'}\| + \|\overline{g_2'g_2}\| =: l_{\text{in}}$. Let $l := \|\overline{og_1}\| + \|\overline{g_1g_2}\|$, then

$$l - l_{in} = \|\overline{og_1}\| - \|\overline{og_1'}\| + \|\overline{g_1g_2'}\| - \|\overline{g_1'g_2'}\|.$$

We have

$$\|\overline{g_1g_2'}\| = \sqrt{4rs+4+\big(sr(r-1)-r/(r-1)\big)^2}, \quad \text{and}$$

$$\|\overline{g_1'g_2'}\| = \sqrt{4rs+4+\big(sr(r-1)-r/(r-1)-1\big)^2}.$$

Hence,

$$\frac{sr(r-1)-r/(r-1)-1/2}{\sqrt{4rs+4+\big(sr(r-1)-r/(r-1)\big)^2}} \leq \|\overline{g_1g_2'}\| - \|\overline{g_1'g_2'}\|.$$

Since $\sqrt{4rs+4+\big(sr(r-1)-r/(r-1)\big)^2} \leq sr(r-1)+r/(r-1)$, we have

$$f_1(r,s) := \frac{sr(r-1)^2-3/2r+1/2}{sr(r-1)^2+r} \leq \|\overline{g_1g_2'}\| - \|\overline{g_1'g_2'}\|.$$

Note that $f_1(r,5) \leq f_1(r,s)$. On the other hand,

$$\|\overline{og_1}\| - \|\overline{og_1'}\| = \sqrt{4rs+4+(r-s)^2} - \sqrt{4rs+4+(r-s+1)^2}.$$

Thus,

$$f_2(r,s) := \frac{s-r-1/2}{\sqrt{4rs+4+(r-s)^2}} \leq \|\overline{og_1}\| - \|\overline{og_1'}\|.$$

If $s = r \geq 5$, then

$$f_1(s,s) \geq f_1(5,5) \geq 0.97 \quad \text{and} \quad f_2(s,s) = \frac{-1}{4\sqrt{s^2+1}} \geq \frac{-1}{4\sqrt{26}}.$$

If $s > r \geq 2$, then

$$f_1(r,s) \geq f_1(2,5) = \frac{7.5}{12} \quad \text{and} \quad f_2(r,s) \geq \frac{1/2}{\sqrt{(r+s)^2+4}} \geq \frac{1}{2\sqrt{53}}.$$

Therefore,

$$l - l_{in} = \|\overline{og_1}\| - \|\overline{og_1'}\| + \|\overline{g_1g_2'}\| - \|\overline{g_1'g_2'}\| \geq f_1(r,s) + f_2(r,s) \geq 0.69 \tag{4.18}$$

Define

$$\epsilon := \frac{2rs - r^2s}{2} + \frac{l}{2} - (r+s),$$

then

$$2\epsilon = \frac{4}{\sqrt{(r+s)^2+4} + (r+s)} + \frac{4(r-1)^2}{\sqrt{\big(s(r-1)^2+r\big)^2 + 4(r-1)^2} + \big(s(r-1)^2+r\big)}.$$

Therefore,

$$2\epsilon \leq \frac{4}{2(r+s)} + \frac{4(r-1)^2}{2\big(s(r-1)^2+r\big)} \leq \frac{2}{r+s} + \frac{2}{s} \leq \frac{2}{7} + \frac{2}{5} \leq 0.686 \tag{4.19}$$

Since $r + s \leq h^0(F)$, Proposition 4.3.4 implies

$$\frac{2rs-r^2s}{2} + \frac{l}{2} - \epsilon \;=\; r+s \;\leq h^0(F) \leq \frac{\chi(F)}{2} + \frac{1}{2}\sum_{i=1}^{n} \|\overline{p_ip_{i-1}}\| \leq \frac{2rs-r^2s}{2} + \frac{l_{\text{in}}}{2}.$$

Therefore, $l - l_{in} \leq 2\epsilon$ this is a contradiction to the inequalities (4.18) and (4.19). Thus the polygon P_F coincides with the triangle T and the morphism $\psi\colon M_{X,H}(\bar{v}) \to T$ is surjective in case (A). □

Corollary 4.4.5. *Let F be a slope-semistable rank r-vector bundle on the curve C of degree $2rs$. Then $h^0(F) \leq r + s$.*

Proof. Using the same notations as in the proof of Proposition 4.4.4, we have

$$h^0(F) \leq \frac{2rs-r^2s}{2} + \frac{1}{2}\sum_{i=1}^{n} \|\overline{p_ip_{i-1}}\| \leq \frac{2rs-r^2s}{2} + \frac{l}{2} = (r+s) + \epsilon < r+s+1.$$

□

4.5 The Brill-Noether loci in the case (B)

In this section, similar to section 4.4, we first show that the morphism $\psi\colon M_{X,H}(\bar{v}) \to$BN described in (4.1) is well-defined in case (B). Then we consider a slope stable rank 4 vector bundle F on the curve C of degree $4p$ and discuss the location of the wall that bounds the Gieseker chamber for i_*F. Finally, we show that the morphism ψ is bijective.

We assume X is a K3 surface with $\mathrm{Pic}(X) = \mathbb{Z}.H$ and $H^2 = 2p$ for some odd number $p \geq 13$. As before, C is any curve in linear system $|H|$ and $i\colon C \hookrightarrow X$ is the embedding of the curve C into the surface X. The push-forward of any slope stable vector bundle $F \in M_C^{\mathrm{st}}(4, 4p)$ which has rank 4 and degree $4p$, has Mukai vector

$$v \coloneqq v(i_*F) = (0, 4H, 0).$$

The moduli space $M_{X,H}(\bar{v})$ of H-Gieseker semistable sheaves on X with Mukai vector

$$\bar{v} \coloneqq (4, 2H, p),$$

is a smooth projective K3 surface [Huy16, Proposition 2.5 and Corollary 3.5]. Let

$$q \coloneqq pr(\bar{v}) = \big(2/p, 4/p\big)\ ,\quad s \coloneqq pr(v - \bar{v}) = \big(-2/p, 4/p\big) \quad \text{and} \quad \tilde{o} = \big(0, 4/p\big).$$

Given a coherent sheaf $E \in M_{X,H}(\bar{v})$, we denote by W_E the cone of the evaluation map as defined in (4.12). Let $s' \coloneqq \big(-1/2, p/4\big)$. Lemma 4.2.5 for $m = 2$, $n = p/2$ and $\epsilon = 1/2$ implies that there is no projection of roots in the grey area and on the open line segment $(\overline{et})$ in Figure 4.12, where $e \coloneqq q_{-m,-n,-\epsilon}$ and $t \coloneqq q'_{-m,-n,-\epsilon}$.

Note that the point t cannot be projection of a root. Indeed, if there exists a root $\delta = (\tilde{r}, \tilde{c}H, \tilde{s}) \in \Delta(X)$ with $pr(\delta) = t$, then

$$\frac{\tilde{c}}{\tilde{s}} = -\frac{2}{5} \quad \text{and} \quad \frac{\tilde{r}}{\tilde{s}} = \frac{p-1}{5},$$

which implies $|\tilde{s}| \geq 5$. Since $\tilde{c}^2p - \tilde{r}\tilde{s} = -1$, we have $\tilde{s}^2(p-5) = 25$ which is impossible for $p > 13$. We also denote by t' the point on the line segment $\overline{s'q}$ with the x-coordinate $-2/5$.

Lemma 4.5.1. *If an object $E \in \mathcal{D}(X)$ with Mukai vector $v(E) = \bar{v}$ is $\sigma_{(0,w)}$-stable for some $w > \sqrt{1/p}$ then it is the shift of an H-Gieseker stable sheaf. Conversely, an H-Gieseker stable sheaf with Mukai vector $v(E) = \bar{v}$ is $\sigma_{(0,w)}$-stable for any $w > \sqrt{1/p}$.*

Proof. By [Bri08, Proposition 14.2], E is $\sigma_{(0,w)}$-stable for $w \gg 0$ precisely if it is the shift of an H-Gieseker stable sheaf. Thus it will be enough to show that there is no wall $\mathcal{W}_E$ intersecting the open line segment $(\overline{oo'})$. Assume for a contradiction that there is a stability condition $\sigma_{(0,w_0)}$ where E is strictly semistable. Up to shift, we may assume $E \in \mathcal{A}(0)$, so there are two $\sigma_{(0,w_0)}$-semistable objects E_1 and E_2 in $\mathcal{A}(0)$ such that they have the same phase and $E_1 \hookrightarrow E \twoheadrightarrow E_2$. By definition, $\mathrm{Im}[Z_{(0,w_0)}(E_i)] = c(E_i) > 0$, hence $c(E_1) = c(E_2) = 1$.

Lemma 4.3.6 for $q_1 = k(0, w_0)$ and $q_2 = pr(\bar{v})$ gives

$$\frac{1}{r(E_1)} = \mu_H\big(H^0(E_1)\big) \geq \mu_H^-\big(H^0(E_1)\big) \geq \frac{1}{2}.$$

Therefore, $r(E_1) \leq 2$ and so $r(E_2) \geq 2$. Since $c(E_2) = 1$, Lemma 4.2.7 implies that E_2

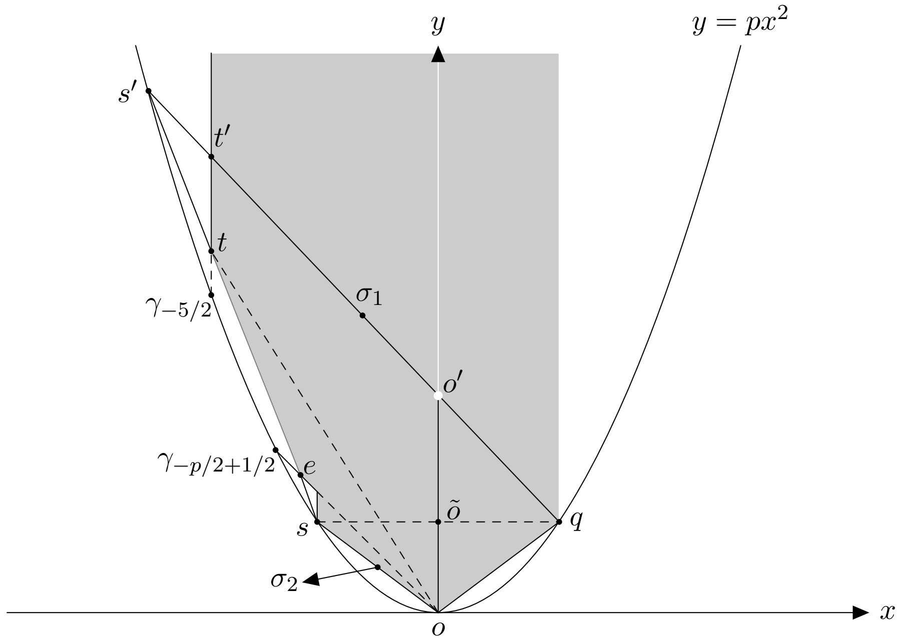

Figure 4.12: No projection of roots in the grey area

is $\sigma_{(0,w)}$-stable for $w > \sqrt{1/p}$ and [MS16, Lemma 6.18] gives $H^{-1}(E_2) = 0$. Now the same argument as above shows $r(E_2) \leq 2$, so $r(E_1) = r(E_2) = 2$. But E_1 and E_2 have the same phase with respect to the stability condition $\sigma_{(0,w_0)}$, thus $s(E_1) = s(E_2) = p/2 \notin \mathbb{Z}$, a contradiction. $\square$

Proposition 4.5.2. *Let $E \in M_{X,H}(\bar{v})$ be an H-Gieseker stable sheaf on X.*

(a) The sheaf E is a μ_H-stable locally free sheaf and the restriction $E|_C$ is a slope-stable vector bundle on the curve C with $h^0(C, E|_C) = 4 + p$. In particular, the morphism ψ described in (4.1) *is well-defined in case (B).*

(b) $Hom_X\big(E, E(-H)[1]\big) = 0$

(c) The object W_E is of the form $W_E = E'[1]$ where E' is a μ_H-stable locally free sheaf on X and $Hom\big(E', E(-H)[1]\big) = 0$.

Proof. Since E is an H-Gieseker stable sheaf, it is also μ_H-semistable. If it is not μ_H-stable, then there are two coherent sheaves E_1 and E_2 with the same slope as $\mu_H(E)$ and $E_1 \hookrightarrow E \twoheadrightarrow E_2$. Therefore, $v(E_1) = (2, H, s_1)$ and $v(E_2) = (2, H, p - s_1)$. Since $v(E_i)^2 \geq -2$ for $i = 1, 2$, we have $s_1 = (p-1)/2$. But, $\dim \mathrm{Ext}^1(E_2, E_1) = \langle v(E_1), v(E_2)\rangle = 0$ which means E is the direct sum of E_1 and E_2, a contradiction. Moreover, μ_H-stability of E implies that its double dual is also μ_H-stable [Huy16, page 156]. Therefore, $v(E^{\vee\vee})^2 \geq -2$ which shows E is a locally free sheaf and proves the first part of (a).

Lemma 4.2.10, part (d) implies that $E(-H)[1]$ is $\sigma_2 \coloneqq \sigma_{(b_2,w_2)}$-stable where $b_2 = -1/2$ and $w_2 \gg 0$. We show that there is no wall $\mathcal{W}_{E(-H)[1]}$ intersecting the closed line segment $[\overline{ot}]$. Assume otherwise. Let F_1 and F_2 are the destabilizing objects on such a wall and $F_1 \hookrightarrow E(-H)[1] \twoheadrightarrow F_2$. Taking cohomology gives a long exact sequence of sheaves

$$0 \to H^{-1}(F_1) \to E(-H) \to H^{-1}(F_2) \xrightarrow{d} H^0(F_1) \to 0.$$

Assume $v\big(H^0(F_1)\big) = (r_1, c_1H, s_1)$ and $v\big(H^{-1}(F_2)\big) = -v(F_2) = (r_2, c_2H, s_2)$. Since $H^{-1}(F_2)$ is a torsion free sheaf, we have $r_2 > 0$ and the surjection $E(-H) \twoheadrightarrow \ker d$ implies $0 \leq r_2 - r_1 \leq 4$.

Assume q_1 and q_2 are the points of intersection of the wall $\mathcal{W}_{E(-H)[1]}$ with the line segments $\overline{oe}$ and $\overline{ot}$, respectively. The slope of the line segment $\overline{oe}$ is equal to the slope of $\overline{o\gamma_{-p/2+1/2}}$, see Figure 4.12. If $r_1 = 0$, then $c_1 \geq 0$ and if $r_1 \neq 0$, then Lemma 4.3.6 for two points q_1 and q_2 implies that

$$\frac{c_2}{r_2} \leq \frac{p-1}{-2p} \quad \text{and} \quad \frac{-2}{p-1} \leq \frac{c_1}{r_1} \qquad \Rightarrow \qquad \frac{c_2}{r_2} < \frac{c_1}{r_1}. \tag{4.20}$$

If $\operatorname{rk}(\ker d) = r_2 - r_1 = 0$, then above inequalities implies $c(\ker d) = c_2 - c_1 < 0$, a contradiction. If $0 < r_2 - r_1 < 4$, then μ_H-stability of $E(-H)$ and inequalities (4.20) give

$$\frac{-1}{2} < \frac{c_2 - c_1}{r_2 - r_1} \leq \frac{c_2}{r_2} \leq \frac{p-1}{-2p}.$$

Therefore,

$$\frac{-1}{2}(r_1 - r_1) < c_2 - c_1 \leq \left(\frac{-1}{2} + \frac{1}{2p}\right)(r_2 - r_1),$$

which is impossible for $c_2 - c_1 \in \mathbb{Z}$ and $p \geq 13$, hence $r_2 - r_1 = 4$. But $H^{-1}(F_1)$ is a torsion free sheaf, so it must be zero and $c_2 - c_1 = -2$. Then inequalities (4.20) imply

$$\left(\frac{p-1}{2}\right)(-c_1) \leq r_1 \leq \left(\frac{2p}{p-1}\right)(-c_1) + \frac{4p}{p-1} - 4.$$

This gives $c_1 = r_1 = 0$, so the subobject F_1 is a skyscraper sheaf. But $E(-H)$ is a locally free sheaf, a contradiction.

Therefore, $E(-H)[1]$ is stable with respect to the stability conditions at the points $\tilde{o}$ and t. Moreover, the structure sheaf $\mathcal{O}_X$ does not make a wall for $E(-H)[1]$ which means $\operatorname{Hom}(\mathcal{O}_X, E(-H)[1]) = 0$. On the other hand, Lemma 4.5.1 implies that E is also stable with respect to the stability condition at the point $\tilde{o}$ and Lemma 4.3.2 shows that $h^0(X, E) = 4 + p$. Now the same argument as in Proposition 4.4.1 implies that $E|_C$ is slope-stable and $h^0(C, E|_C) = 4 + p$. Moreover, $\operatorname{Hom}(E, E(-H)[1]) = 0$. This completes the proof of (a) and (b).

Let σ_1 be a stability condition on the line segment $\overline{o't'}$ and sufficiently close to the point o'. Since the structure sheaf $\mathcal{O}_X$ and E are σ_1-semistable of the same phase, the co-kernel W_E of the evaluation map is also σ_1-semistable. If it is not σ_1-stable, then it has a subobject E_1 with Mukai vector $v(E_1) = t_1 v(E) + s_1 v(\mathcal{O}_X)$ where $0 \leq t_1 \leq 1$. Moreover, $c(E_1) = t_1 c(E) = 2t_1 \in \mathbb{Z}$. If $t_1 = 1/2$, then

$$\frac{1}{2}(4, 2, p) + s_i(1, 0, 1) \notin \mathbb{Z}^3.$$

Therefore, $t_i = 0, 1$ which means $\mathcal{O}_X$ is either a subobject or a quotient of W_E. But $\operatorname{Hom}(W_E, \mathcal{O}_X) = \operatorname{Hom}(\mathcal{O}_X, W_E) = 0$, a contradiction. Hence, W_E is σ_1-stable.

If the stability condition $\sigma_{(b,w)}$ is on the line segment $\overline{ot'}$, then $b = -1/(p/2 + 1/2)$, see Figure 4.13. Therefore, Lemma 4.2.7 implies that there is no wall for W_E intersecting the open line segment $(\overline{ot'})$, so it is $\sigma_{(0,w)}$-stable where $w \gg 0$. Then [MS16, Lemma 6.18] shows that $W_E \cong E'[1]$ where E' is a μ_H-stable locally free sheaf. Hence $E'[1]$ is stable with respect to the stability conditions on the line segment $\overline{s'o}$, by Lemma 4.2.10, part (d). Moreover, if $\sigma_{(b,w)}$ is on the line segment $\overline{ot}$, then

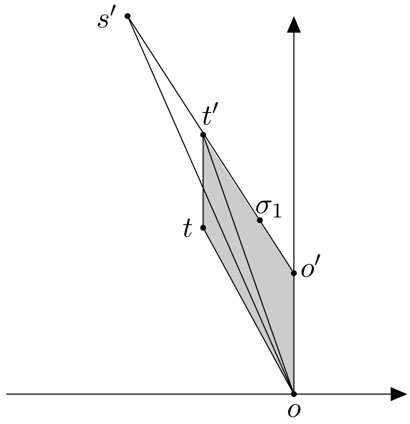

Figure 4.13: No projection of roots in the grey area and at the point t

$b = -1/(p/2 - 1/2)$ and Lemma 4.2.7 implies that there is no wall for E' intersecting the line segment $\overline{ot}$. Therefore, E' is stable with respect to the stability condition at the point t and it has also the same phase as $E(-H)[1]$, see Figure 4.12. So there is no non-trivial homomorphism between them, which completes the proof of (c). □

4.5.1 The first wall

Lemma 4.2.10, part (a) implies that the push-forward of any slope-stable vector bundle $F \in M_C^{\mathrm{st}}(4, 4p)$ is stable with respect to the stability conditions in the Gieseker chamber. The next proposition describes the wall that bounds the Gieseker chamber for i_*F. Note that the walls for i_*F are horizontal line segments, by Proposition 4.2.6, part (d).

Proposition 4.5.3. *Let $F \in M_C^{st}(4, 4p)$ be a slope-stable vector bundle on the curve C, then the wall that bounds the Gieseker chamber for i_*F is not below the line segment $\overline{sq}$ and it coincides with the line segment $\overline{sq}$ if and only if F is the restriction of a vector bundle $E \in M_{X,H}(\bar{v})$ to the curve C.*

Proof. Assume the wall $\mathcal{W}_{i_*F}$ that bounds the Gieseker chamber for i_*F is below or on the line segment $\overline{sq}$, see Figure 4.14.

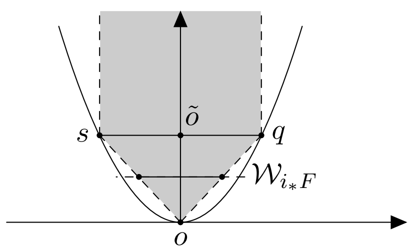

Figure 4.14: The first wall $\mathcal{W}_{i_*F}$

Assume the stability condition $\sigma_{(0,w^*)}$ is on the wall $\mathcal{W}_{i_*F}$. There is a destabilising sequence $F_1 \hookrightarrow i_*F \twoheadrightarrow F_2$ of objects in $\mathcal{A}(0)$ such that F_1 and F_2 are $\sigma_{(0,w^*)}$-semistable objects of the same phase as i_*F and $\phi_{(0,w)}(F_1) > \phi_{(0,w)}(i_*F)$ for $w < w_*$. We may assume F_1 is $\sigma_{(0,w^*)}$-stable. Taking cohomology gives a long exact sequence of sheaves

$$0 \to H^{-1}(F_2) \to F_1 \xrightarrow{d_0} i_*F \to H^0(F_2) \to 0. \tag{4.21}$$

Let $v(F_1) = \big(r', c'H, s'\big)$, $v\big(H^0(F_2)\big) = \big(0, c''H, s''\big)$ and $v(T(F_1)) = (0, \tilde{r}H, \tilde{s})$, where $T(F_1)$ is the maximal torsion subsheaf of F_1. Using Lemma 4.3.6, in the same way as in the proof of Proposition 4.4.2, gives $0 < r' = 4 - c'' - \tilde{r}$ and both sheaves $F_1/T(F_1)$ and $H^{-1}(F_2)$ are μ_H-semistable. Moreover,

$$\mu_H\big(F_1/T(F_1)\big) = \frac{c' - \tilde{r}}{4 - c'' - \tilde{r}} = \frac{1}{2},$$

which gives $c' - \tilde{r} = 1$ or 2.

If $c' - \tilde{r} = 1$, then $c'' + \tilde{r} = 2$ and $r' = 2$. Since we assumed the wall $\mathcal{W}_{i_*F}$ is below or on the line segment $\overline{sq}$, the point $pr(v(F_1))$ is not above the line $\overline{sq}$, so

$$\frac{r'}{s'} = \frac{2}{s'} \leq \frac{4}{p} \quad \Rightarrow \quad s' \geq \frac{p}{2}. \tag{4.22}$$

Since $H^0(F_2)$ is a quotient of i_*F in $\mathrm{Coh}(X)$, the H-Gieseker stability of i_*F gives $s'' \geq 0$ and if $c'' \neq 0$, then $s'' > 0$. Also, μ_H-stability of $H^{-1}(F_2)$ implies

$$v\big(H^{-1}(F_2)\big)^2 = \big(2, -H, s' + s''\big)^2 \geq -2 \quad \Rightarrow \quad \frac{p+1}{2} \geq s' + s''.$$

Therefore, $s'' = 0 = c''$ and $\tilde{r} = 2$ which implies $T(F_1) \neq 0$. There is a short exact sequence of coherent sheaves

$$T(F_1) \hookrightarrow F_1 \twoheadrightarrow F_1/T(F_1)$$

in $\mathcal{A}(0)$ where $F_1/T(F_1)$ is the torsion free part of F_1. Since F_1 is $\sigma_{(0,w^*)}$-stable, we have

$$\phi_{(0,w^*)}(T(F_1)) < \phi_{(0,w^*)}(F_1) = \phi_{(0,w^*)}(i_*F)$$

which implies $\tilde{s} < 0$. On the other hand, the sheaf $F_1/T(F_1)$ is μ_H-semistable, so

$$v\big(F_1/T(F_1)\big)^2 = \big(2, H, -\tilde{s} + \frac{p+1}{2}\big)^2 \geq -2 \quad \Rightarrow \quad \tilde{s} \geq 0$$

which is a contradiction.

Therefore, $c' - \tilde{r} = 2$ and $r' = 4 - c'' - \tilde{r} = 4$, hence

$$c'' = \tilde{r} = 0 \quad \text{and} \quad v(F_1) = \big(4, 2H, s'\big).$$

By assumption, the point $pr(v(F_1))$ is below or on the line segment $\overline{sq}$, so $s' \geq p$. Since F_2 is $\sigma_{(0,w^*)}$-stable, we have $v(F_2)^2 = 8p - 8s' \geq -2$, which gives $s' = p$. Therefore, the wall $\mathcal{W}_{i_*F}$ cannot be below the line segment $\overline{sq}$.

Moreover, Lemma 4.5.1 and Proposition 4.5.2 imply that F_1 is a μ_H-stable locally free sheaf and $F_1|_C$ is slope-stable. The non-zero morphism d_0 in the long exact sequence (4.21) factors via the morphism $d_0'\colon i_*F_1|_C \to i_*F$. The objects $i_*F_1|_C$ and i_*F have the same Mukai vector and so have the same phase and $i_*F_1|_C$ is H-Gieseker stable. Thus, the morphism d_0' must be an isomorphism and $F \cong F_1|_C$. $\square$

4.5.2 The maximum number of global sections

Given a vector bundle F in the moduli space $M_C^{\mathrm{st}}(4, 4p)$, Proposition 4.3.4 implies that there is a positive real number w' such that the HN filtration of i_*F is a fixed sequence

$$0 = \tilde{E}_0 \subset \tilde{E}_1 \subset .... \subset \tilde{E}_{n-1} \subset \tilde{E}_n = i_*F,$$

for all stability conditions of form $\sigma_{(0,w)}$ where $\sqrt{1/p} < w < w'$. By applying the same argument as in Lemma 4.4.3, one can show that the polygon P_F with the extremal points

$$p_i = \overline{Z}(\tilde{E}_i) \coloneqq Z_{(0,\sqrt{1/p})}(\tilde{E}_i) \quad \text{for } \ 0 \leq i \leq n,$$

is inside the triangle T with the vertices $g_1 \coloneqq \overline{Z}(\bar{v}) = 4 - p + 2i$, $g_2 \coloneqq \overline{Z}(F) = 4i$ and the origin.

Proposition 4.5.4. *Let F be a slope-stable rank 4 vector bundle on the curve C of degree $4p$. If $h^0(F) \geq 4 + p$, then F is the restriction of a vector bundle $E \in M_{X,H}(\bar{v})$ to the curve C. In particular, the morphism ψ described in (4.1) is bijective in case (B).*

Proof. We first show that the polygon P_F coincides with the triangle $T = \triangle o g_1 g_2$. Assume for a contradiction that the polygon P_F is strictly inside the triangle T, so it is contained in the polygon $o s_1 s_2 s_3 g_2$ where $s_1 \coloneqq 2 - p/2 + i$, $s_2 \coloneqq 5 - p + 2i$, $s_3 \coloneqq 2 - p/2 + 3i$, see Figure 4.15. Therefore,

$$\sum_{i=1}^{i=n} \|\overline{p_i p_{i-1}}\| \leq \|\overline{o s_1}\| + \|\overline{s_1 s_2}\| + \|\overline{s_2 s_3}\| + \|\overline{s_3 g_2}\| =: l_{in}.$$

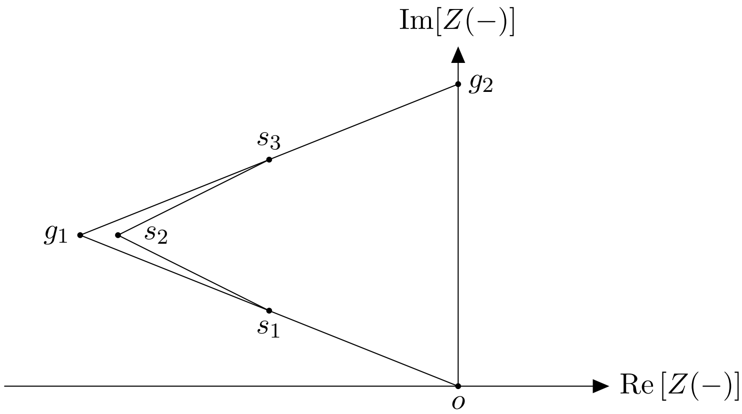

Figure 4.15: HN-polygon of a vector bundle $F \in T$

Let $l \coloneqq \|\overline{og_1}\| + \|\overline{g_1 g_2}\|$ and $\epsilon \coloneqq \dfrac{l}{2} - (p+4)$. Proposition 4.3.4 implies

$$\frac{l}{2} - \epsilon = 4 + p \leq h^0(F) \leq \frac{l_{in}}{2}$$

Therefore

$$l - l_{in} = \frac{4(p-5)}{\sqrt{(p+4)^2 + 16} + \sqrt{(p+2)^2 + 48}} \ \leq \ 2\epsilon = \frac{32}{\sqrt{(p+4)^2 + 16} + p + 4},$$

which is impossible for $p \geq 13$. Thus, the polygon P_F coincides with the triangle T.

If $v(\tilde{E}_1) = (r_1, c_1H, s_1)$, then $c_1 = 1$ or 2. If $c_1 = 1$, then $s_1 - r_1 = p/2 - 2 \notin \mathbb{Z}$, a contradiction. Therefore, $c_1 = 2$ and $v(\tilde{E}_1) = (4-k, 2H, p-k)$. Since $\tilde{E}_1$ is $\sigma_{(0,w)}$-semistable for $w > w'$, Lemma 4.3.1 gives $v(\tilde{E}_1)^2 = 8p - 2(4-k)(p-k) \geq -8$, so $0 \leq k \leq 4$. Moreover, Proposition 4.5.3 implies that the point $pr(v(\tilde{E}_1)) = \big(2/(p-k)\,,\,(4-k)/(p-k)\big)$ is above or on the line segment $\overline{sq}$, hence

$$\frac{4-k}{p-k} \geq \frac{4}{p},$$

which gives $k = 0$ for $p \geq 13$. Therefore, $(\tilde{E}_1) = \bar{v}$ and F is the restriction of the vector bundle $\tilde{E}_1 \in M_{X,H}(\bar{v})$, by Proposition 4.5.3. This implies the morphism ψ is surjective in case (B) and the injectivity comes from the uniqueness of the Harder-Narasimhan filtration. □

Corollary 4.5.5. *Let F be a slope-stable rank 4 vector bundle on the curve C of degree $4p$. Then $h^0(F) \leq 4 + p$.*

Proof. Using the same notations as in the proof of Proposition 4.5.4, we have

$$h^0(F) \leq \frac{1}{2}\sum_{i=1}^{i=n} \|\overline{p_i p_{i-1}}\| \leq \frac{l}{2} = p + 4 + \epsilon < p + 4 + 1.$$

□

4.6 The final results

In this section, we assume that the moduli spaces $M_{X,H}(\bar{v})$ and BN are defined either as in the case (A) or (B) and prove the main results.

Proof of Theorem 4.1.2. By Proposition 4.4.4, the morphism $\psi\colon M_{X,H}(\bar{v}) \to$BN is bijective in case (A) and Proposition 4.4.1 implies that any vector bundle F in the Brill-Noether locus BN$= M_C(r, 2rs, r+s)$ is slope-stable and $h^0(F) = r+s$. Moreover, Propositions 4.5.4 and 4.5.2 show that the morphism ψ is bijective in case (B) and any vector bundle F in the Brill-Noether locus BN$= M_C^{st}(4, 4p, p+4)$ is slope-stable and $h^0(F) = 4 + p$. Hence we only need to show that the morphism ψ induces an isomorphism of tangent spaces. The Zariski tangent space to the Brill-Noether locus BN at the point $[F]$ is the kernel of the map

$$k_1\colon \operatorname{Ext}^1(F,F) \to \operatorname{Hom}\big(H^0(C,F), H^1(C,F)\big),$$

where any $f\colon F \to F[1] \in \operatorname{Ext}^1(F,F) = \operatorname{Hom}_C(F, F[1])$ goes to

$$k_1(f) = H^0(f)\colon \operatorname{Hom}_C(\mathcal{O}_C, F) \to \operatorname{Hom}_C(\mathcal{O}_C, F[1]),$$

see [BS13, Proposition 4.3] for details. Note that the proof in this paper is valid for any family of simple sheaves on a variety. In addition, for any vector bundle E in the moduli space $M_{X,H}(\bar{v})$,

$$T_{[E]}\big(M_{X,H}(\bar{v})\big) = \operatorname{Hom}_X(E, E[1]).$$

Let $i\colon C \hookrightarrow X$ be the closed embedding of the curve C into the surface X, then $Ri_*(-) = i_*(-)$ and for a vector bundle E on X, we have $Li^*(E) = i^*(E)$. The derivative of the restriction map

$$d\psi\colon T_{[E]}\big(M_{X,H}(\bar{v})\big) \to T_{[E|_C]}\mathrm{BN},$$

sends any $f\colon E \to E[1] \in \mathrm{Hom}_X(E, E[1])$ to its restriction $i^*f\colon i^*E \to i^*E[1] \in \ker(k_1)$.

Define $h\colon \mathrm{id}_{\mathcal{D}(X)} \to Ri_*Li^*$ as the natural transformation for the pair of adjoint functors $Li^* \dashv Ri_*$. Given a vector bundle E in the moduli space $M_{X,H}(\bar{v})$ and a morphism $\varphi \in \mathrm{Hom}_X(E, E[1])$, we have the commutative diagram

$$\begin{tikzcd} E \arrow[r, "h_E"] \arrow[d, "\varphi"'] & i_*i^*E \arrow[d, "i_*i^*\varphi"] \\ E[1] \arrow[r, "h_{E[1]}"] & i_*i^*E[1]. \end{tikzcd} \tag{4.23}$$

Therefore the following diagram is also commutative

$$\begin{tikzcd} \mathrm{Hom}_X\big(E, E[1]\big) \arrow[r, "d\psi"] \arrow[dr, "k_2 := h_{E[1]}\circ(-)"'] & \mathrm{Hom}_C\big(i^*E, i^*E[1]\big) \arrow[r, "k_1"] \arrow[d, "\sim"', "i_*(-)\circ h_E =: k_3"] & \mathrm{Hom}\big(H^0(C, i^*E), H^1(C, i^*E)\big). \\ & \mathrm{Hom}_X\big(E, i_*i^*E[1]\big) & \end{tikzcd}$$

Applying the functor $\mathrm{Hom}_X(E, -)$ to the distinguished triangle

$$E[1] \xrightarrow{h_{E[1]}} i_*i^*E[1] \xrightarrow{g} E(-H)[2]$$

gives the long exact sequence

$$\ldots \to \mathrm{Hom}_X\big(E, E(-H)[1]\big) \to \mathrm{Hom}_X\big(E, E[1]\big) \xrightarrow{k_2} \mathrm{Hom}_X\big(E, i_*i^*E[1]\big) \to \ldots$$

By Propositions 4.4.1 and 4.5.2, $\mathrm{Hom}_X\big(E, E(-H)[1]\big) = 0$, hence the morphism k_2 is injective which implies the derivative $d\psi$ is injective. Moreover, Propositions 4.4.1 and 4.5.2 imply that we have the short exact sequence

$$0 \to E' \to \mathcal{O}_X^{h^0(E)} \xrightarrow{\mathrm{ev}_E} E \to 0$$

in $\mathrm{Coh}(X)$. Since $\mathrm{Hom}_X\big(E', E(-H)[1]\big) = 0$, applying the functor $\mathrm{Hom}_X\big(-, E(-H)[2]\big)$ gives the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_X\big(E, E(-H)[2]\big) \xrightarrow{\Phi} \mathrm{Hom}_X\big(&\mathcal{O}_X^{h^0(E)}, E(-H)[2]\big) \to \\ &\mathrm{Hom}_X\big(E', E(-H)[2]\big) \to 0. \end{aligned} \tag{4.24}$$

Assume $\xi \in \mathrm{Ker}(k_1) \subseteq \mathrm{Hom}_C\big(i^*E, i^*E[1]\big)$, then for any morphism $f\colon i^*\mathcal{O}_X \to i^*E$, the composition

$$\xi \circ f\colon i^*\mathcal{O}_X \to i^*E \to i^*E[1]$$

vanishes. Consider the morphism $h_E \circ \mathrm{ev}_E\colon \mathcal{O}_X^{h^0(E)} \to i_*i^*E$, then the composition

$$i_*\xi \circ (h_E \circ \mathrm{ev}_E)\colon \mathcal{O}_X^{h^0(E)} \to i_*i^*E \to i_*i^*E[1]$$

vanishes by adjunction. Thus, $\Phi\big(g\circ i_*\xi\circ h_E\big)=0$ and the exact sequence (4.24) implies $g\circ i_*\xi\circ h_E=0$. Therefore there exist morphisms ξ' and ξ'' such that the following diagram commutes.

$$\begin{tikzcd} E \arrow[r, "h_E"] \arrow[d, dashed, "\exists\,\xi'"'] & i_*i^*E \arrow[r] \arrow[d, "i_*\xi"'] & E(-H)[1] \arrow[d, dashed, "\exists\,\xi''"'] \\ E[1] \arrow[r, "h_{E[1]}"] & i_*i^*E[1] \arrow[r, "g"] & E(-H)[2] \end{tikzcd}$$

This implies $i_*\xi\circ h_E=h_{E[1]}\circ\xi'$. Moreover, the commutative diagram (4.23) gives $h_{E[1]}\circ\xi'=i_*i^*\xi'\circ h_E$. Therefore, the morphisms $i_*\xi\circ h_E$ and $i_*i^*\xi'\circ h_E$ are equal in $\mathrm{Hom}_X\big(E,i_*i^*E[1]\big)$. Finally, the isomorphism k_3 implies that $\xi=i^*\xi'$ which shows $d\psi$ is surjective. This completes the proof of Theorem 4.1.2. $\square$

Proof of Theorem 4.1.3. Let (X,H) be a polarised K3 surface with $\mathrm{Pic}(X)=\mathbb{Z}.H$, and let C be any curve in the linear system $|H|$. The moduli space $N:=M_{X,H}(\bar{v})$ which is defined either as in case (A) or (B) is a smooth projective K3 surface. Moreover, there exists a Brauer class $\alpha\in\mathrm{Br}(N)$ and a universal $(1\times\alpha)$-twisted sheaf $\tilde{\mathcal{E}}$ on $X\times(N,\alpha)$.

Theorem 4.1.2 implies that the moduli space N is isomorphic to the Brill-Noether locus BN and the restriction of the universal twisted sheaf $\tilde{\mathcal{E}}|_{C\times(\mathrm{BN},\alpha)}$ is a universal $(1\times\alpha)$-twisted sheaf on $C\times(\mathrm{BN},\alpha)$, so $v'=v\big(\tilde{\mathcal{E}}|_{p\times(\mathrm{BN},\alpha)}\big)$ for a point p on the curve C.

Let H' be a generic polarisation on N. Then the moduli space $M_{(N,\alpha),H'}(v')$ of α-twisted semistable sheaves on N with respect to H', is isomorphic to the original K3 surface X (see e.g. [Yos15, Theorem 2.7.1]). Therefore, $M_{(\mathrm{BN},\alpha),H'}(v')\cong X$ which completes the proof of Theorem 4.1.3. $\square$

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